JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **25**, No. 1, pp. 41-65

doi: 10.4208/jpde.v25.n1.4
February 2012

Partial Differential Equations that are Hard to Classify

HOWISON S. D.[1], LACEY A. A.[2,*] and OCKENDON J. R.[1]

[1] *OCIAM, Mathematical Institute, University of Oxford, 24-29 St. Giles', Oxford, OX1 3LB, UK.*
[2] *Maxwell Institute for Mathematical Sciences, and School of Mathematical and Computer Sciences, Heriot-Watt University, Riccarton, Edinburgh, EH14 4AS, UK.*

Received 10 June 2011; Accepted 24 November 2011

Abstract. Semi-linear $n\times n$ systems of the form $\mathbf{A}\partial\mathbf{u}/\partial x+\mathbf{B}\partial\mathbf{u}/\partial y=\mathbf{f}$ can generally be solved, at least locally, provided data are imposed on non-characteristic curves. There are at most n characteristic curves and they are determined by the coefficient matrices on the left-hand sides of the equations. We consider cases where such problems become degenerate as a result of ambiguity associated with the definition of characteristic curves. In such cases, the existence of solutions requires restrictions on the data and solutions might not be unique.

AMS Subject Classifications: 35A21, 35A30, 35E20, 35N05

Chinese Library Classifications: O175.2

Key Words: Linear systems of first-order PDEs; classification; canonical systems.

---

1 Introduction

It is well known that the Cauchy-Kowalevski Theorem tells us that a problem of the form

$$\mathbf{A}\frac{\partial\mathbf{u}}{\partial x}+\mathbf{B}\frac{\partial\mathbf{u}}{\partial y}=\mathbf{f}, \tag{1.1}$$

where $\mathbf{u}$ is an n-dimensional vector and $\mathbf{A}$ and $\mathbf{B}$ are $n\times n$ constant matrices, has an analytic solution, at least locally, provided we have analytic data on a non-characteristic analytic curve. The unique solution can be determined, locally, by solving n scalar equations given by (1.1), in conjunction with the n found by differentiating the Cauchy data

$$\mathbf{u}=\mathbf{U}_0(t) \qquad \text{on } \mathbf{x}=\mathbf{x}_0(t) \tag{1.2}$$

*Corresponding author. *Email addresses:* `howison@maths.ox.ac.uk` (S. D. Howison), `A.A.Lacey@hw.ac.uk` (A. A. Lacey), `ock@maths.ox.ac.uk` (J. R. Ockendon)

along the curve $(x,y)=\mathbf{x}=\mathbf{x}_0(t)=(x_0(t),y_0(t))$, to find the $2n$ first partial derivatives $\partial\mathbf{u}/\partial x$ and $\partial\mathbf{u}/\partial y$. An entirely equivalent way of thinking about characteristics is to regard them as curves across which $\mathbf{u}$ can have discontinuous first derivatives.

The Cauchy-Kowalevski argument fails when the curve is characteristic so that

$$\lambda=\frac{\mathrm{d}x}{\mathrm{d}t},\qquad \mu=\frac{\mathrm{d}y}{\mathrm{d}t}\quad \text{(not both zero)} \tag{1.3}$$

are such that (1.1) together with the equations got from differentiating (1.2), in vector form

$$\lambda\frac{\partial\mathbf{u}}{\partial x}+\mu\frac{\partial\mathbf{u}}{\partial y}=\mathbf{U}_0', \tag{1.4}$$

fail to have a unique solution. This of course happens with λ, μ such that

$$\left|\begin{array}{c|c}\mathbf{A} & \mathbf{B}\\ \hline \lambda\mathbf{I} & \mu\mathbf{I}\end{array}\right|=\left|\begin{array}{cccccc} a_{11} & \cdots & a_{1n} & b_{11} & \cdots & b_{1n}\\ \vdots & & \vdots & \vdots & & \vdots\\ a_{n1} & \cdots & a_{nn} & b_{n1} & \cdots & b_{nn}\\ \lambda & \cdots & 0 & \mu & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda & 0 & \cdots & \mu \end{array}\right|=0, \tag{1.5}$$

where $\mathbf{I}$ is the $n\times n$ identity matrix. Equivalently,

$$\left|\begin{array}{cccccc} \mu a_{11}-\lambda b_{11} & \cdots & \mu a_{1n}-\lambda b_{1n} & b_{11} & \cdots & b_{1n}\\ \vdots & & \vdots & \vdots & & \vdots\\ \mu a_{n1}-\lambda b_{n1} & \cdots & \mu a_{nn}-\lambda b_{nn} & b_{n1} & \cdots & b_{nn}\\ 0 & \cdots & 0 & 1 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{array}\right|=|\mu\mathbf{A}-\lambda\mathbf{B}|=0. \tag{1.6}$$

For "most" problems, with no sort of degeneracy associated with the left-hand side of (1.1), the condition (1.5) would make the curve direction (λ,μ) that of the characteristic.

In the present paper we consider problems such that (1.5) holds for <u>all</u> λ, μ, so that, whatever direction is used, the system (1.1) fails to have a unique solution. We anticipate that, since the coefficient matrix of the combined system

$$\left(\begin{array}{c|c}\mathbf{A} & \mathbf{B}\\ \hline \lambda\mathbf{I} & \mu\mathbf{I}\end{array}\right)\left(\begin{array}{c}\partial\mathbf{u}/\partial x\\ \hline \partial\mathbf{u}/\partial y\end{array}\right)=\left(\begin{array}{c}\mathbf{f}\\ \mathbf{U}_0'\end{array}\right) \tag{1.7}$$

is singular, whatever data curve is chosen, at least one compatibility condition relating $\mathbf{f}$ and $\mathbf{u}_0$ has to be satisfied if the problem (1.1), (1.2) is to have a solution; moreover, that if this condition holds, the problem can have multiple solutions. It is clear that degeneracy is associated with the rank of $\mu\mathbf{A}-\lambda\mathbf{B}$ being identically less than n.