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On the Steady Solutions to a Model of Compressible Heat Conducting Fluid in Two Space Dimensions

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Abstract. We consider steady compressible Navier–Stokes–Fourier system in a bounded two-dimensional domain with the pressure law $p(\varrho,\vartheta)\sim\varrho\vartheta+\varrho\ln^{\alpha}(1+\varrho)$. For the heat flux $\mathbf{q}\sim-(1+\vartheta^m)\nabla\vartheta$ we show the existence of a weak solution provided $\alpha>\max\{1,1/m\}$, $m>0$. This improves the recent result from [1].

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1 Introduction, main result

We consider the following system of partial differential equations

$$\operatorname{div}(\varrho\mathbf{u})=0, \tag{1.1}$$

$$\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u})-\operatorname{div}\mathbb{S}+\nabla p=\varrho\mathbf{f}, \tag{1.2}$$

$$\operatorname{div}(\varrho E\mathbf{u})=\varrho\mathbf{f}\cdot\mathbf{u}-\operatorname{div}(p\mathbf{u})+\operatorname{div}(\mathbb{S}\mathbf{u})-\operatorname{div}\mathbf{q}. \tag{1.3}$$

It is a well-known model for steady flow of a compressible heat conducting fluid. Here, ϱ is the density of the fluid, $\mathbf{u}$ is the velocity field, $\mathbb{S}$ the viscous part of the stress tensor, p the pressure, $\mathbf{f}$ the external force, E the specific total energy and $\mathbf{q}$ the heat flux. We consider system (1.1)–(1.3) in a bounded domain $\Omega\subset\mathbb{R}^2$. At the boundary $\partial\Omega$ we assume the boundary conditions

$$\mathbf{u}=\mathbf{0}, \tag{1.4}$$

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$$-\mathbf{q}\cdot\mathbf{n}+L(\vartheta-\Theta_0)=0, \tag{1.5}$$

with $\mathbf{n}$ the outer normal to $\partial\Omega$, $L=\text{const}>0$ and $\Theta_0=\Theta_0(x)>0$, both given. Furthermore, the total mass of the fluid

$$\int_\Omega \varrho \, \mathrm{d}x = M > 0 \tag{1.6}$$

is also given.

We have to specify the constitutive relations for the quantities $\mathbb{S}$, p, E and $\mathbf{q}$. The fluid is assumed to be newtonian, i.e. we have

$$\mathbb{S}=\mathbb{S}(\vartheta,\mathbf{u})=\mu(\vartheta)\Big[\nabla\mathbf{u}+(\nabla\mathbf{u})^T-\operatorname{div}\mathbf{u}\mathbb{I}\Big]+\xi(\vartheta)\operatorname{div}\mathbf{u}\mathbb{I}, \tag{1.7}$$

with viscosity coefficients $\mu(\vartheta)$ and $\xi(\vartheta)$. In our paper, we consider the viscosity coefficients to be given globally Lipschitz functions of the temperature ϑ such that

$$c_1(1+\vartheta)\le\mu(\vartheta)\le c_2(1+\vartheta), \qquad 0\le\xi(\vartheta)\le c_2(1+\vartheta). \tag{1.8}$$

The heat flux satisfies the Fourier law

$$\mathbf{q}=\mathbf{q}(\vartheta,\nabla\vartheta)=-\kappa(\vartheta)\nabla\vartheta, \tag{1.9}$$

with $\kappa(\cdot)\in C([0,\infty))$ such that for a certain $m>0$

$$c_3(1+\vartheta^m)\le\kappa(\vartheta)\le c_4(1+\vartheta^m). \tag{1.10}$$

The specific total energy E has the form

$$E=E(\varrho,\vartheta,\mathbf{u})=\frac{1}{2}|\mathbf{u}|^2+e(\varrho,\vartheta), \tag{1.11}$$

where e stands for the specific internal energy; we will specify this quantity below.

We consider the pressure law of the form

$$p=p(\varrho,\vartheta)=\varrho\vartheta+\frac{\varrho^2}{\varrho+1}\ln^\alpha(1+\varrho) \tag{1.12}$$

with $\alpha>0$. In agreement with the second law of thermodynamics, there exists specific entropy, a function of ϱ and ϑ, given up to an additive constant by the Gibbs relation

$$\frac{1}{\vartheta}\Big(De(\varrho,\vartheta)+p(\varrho,\vartheta)D\Big(\frac{1}{\varrho}\Big)\Big)=Ds(\varrho,\vartheta). \tag{1.13}$$

The specific entropy, due to (1.2) and (1.3), fulfills

$$\operatorname{div}(\varrho s\mathbf{u})+\operatorname{div}\Big(\frac{\mathbf{q}}{\vartheta}\Big)=\frac{\mathbb{S}:\nabla\mathbf{u}}{\vartheta}-\frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2} \tag{1.14}$$