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On the Spectrum of a Class of Strongly Coupled p-Laplacian Systems

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Abstract. Consider the nonlinear coupled elliptic system

$$\begin{cases} -\Delta_p u - \lambda V(x)|u|^\alpha |v|^\beta v = \mu |u|^\alpha |v|^\beta v & \text{in } \Omega, \\ -\Delta_q v - \lambda V(x)|u|^\alpha |v|^\beta u = \mu |u|^\alpha |v|^\beta u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_\rho \xi = \nabla \cdot (|\nabla \xi|^{\rho-2} \nabla \xi)$, $\rho > 1$, Ω is a bounded domain and $V(x)$ is a potential weight function. We prove that for any real parameter λ, there is at least a sequence of eigencurves $(\mu_k(\lambda))_k$ by using an energy variational method. We prove also via an homogeneity type condition that the eigenvector corresponding to the principal frequency $\mu_1(\lambda)$ is unique *modulo scaling*, bounded, regular and positive, without any condition on regularity of the domain. We end this work by giving a new proof technique to prove the simplicity of $\mu_1(\lambda)$ via a new version of Picones' identity.

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1 Introduction

Let Ω be a bounded domain (i.e., open and connected set) in $\mathbb{R}^N$ not necessary regular; $N > 1, p > 1, q > 1$, and $\alpha, \beta > -1$ satisfying the homogeneity type assumption:

$$(\mathcal{H}) \qquad \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1,$$

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and $V(x)\in L^{\infty}(\Omega)\setminus\{0\}$ be an indefinite weight function which can change the sign. Here, λ is a real parameter and μ plays the role of spectral parameter (eigenvalue). We consider the following nonlinear elliptic system

$$(\mathcal{S}_{\lambda})\qquad \begin{cases} -\Delta_p u-\lambda V(x)|u|^{\alpha}|v|^{\beta}v=\mu|u|^{\alpha}|v|^{\beta}v & \text{in } \Omega,\\ -\Delta_q v-\lambda V(x)|u|^{\alpha}|v|^{\beta}u=\mu|u|^{\alpha}|v|^{\beta}u & \text{in } \Omega,\\ u=v=0, & \text{on } \partial\Omega. \end{cases}$$

Note that the system (S_{λ}) is of two nonlinear second-order elliptic equations. It is strongly coupled in the sense that the interaction is present in right-hand side product terms as in the left-hand side source terms.

The differential operator involved is $\Delta_{\rho}\xi=\nabla\cdot(|\nabla\xi|^{\rho-2}\nabla\xi)$, which reduces to famous Laplace operator Δ, when $\rho=2$. We mention that problem related to p-Laplacian arise both from pure and applied mathematics. For instance, Like in the theory of quasi-regular and quasi-conformal mapping, as well as from a variety of applications, e.g. Non-Newtonian fluids, reaction diffusion problems, flow through porous media. Non-linear elasticity, glaciology, petroleum extraction, astronomy,$\cdots$etc. We can cite for more details, [1] or [2] and the references therein.

Several special cases of system S_{λ} have been considered in literature. For the case of scalar equation, that is, S_{λ} reduce to single equation, when $p=q$ and $\alpha\cdot\beta=0$, replacing the system

$$-\Delta_p w-\lambda V(x)|w|^{p-2}w=\mu|w|^{p-2}w.$$

This case is largely considered by various works, we cite for example [3–9].

Concerning systems of the type (S_{λ}), (but weak coupled), a lot of papers have appeared in recent years dealing with equations involving p-Laplacian both in bounded and unbounded domains. In particular de Thelin in [5] obtained the simplicity of the first eigenvalue of (S_0) by considering smooth bounded domains. The study of systems of the type (S_{λ}) in the whole $\mathbb{R}^N$ was continued in [8], where the authors considered one spectral parameter and under some large hypothesis on the weight functions in the constructive system of the form

$$\begin{aligned} &\Delta_p u=\lambda a(x)|u|^{p-2}u+\lambda b(x)|u|^{\alpha-1}|v|^{\beta+1}u && \text{in } \mathbb{R}^N,\\ &\Delta_q v=\lambda d(x)|v|^{q-2}v+\lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v && \text{in } \mathbb{R}^N,\\ &\lim_{|x|\to+\infty}u(x)=\lim_{|x|\to+\infty}v(x)=0. \end{aligned}$$

Later, in [10], the author showed the simplicity of the principal eigenvalue of the last system with $\lambda=0$, by extending the Díaz-Sáa's inequality to the whole $\mathbb{R}^N$.

It is important to indicate that El Khalil et al. studied in [11] the stability with respect to the rheological exponent p and q of the first eigenvalue ($\mu_1(0)=:\lambda_1$) of (S_0), where the domain Ω satisfying the geometrical regularity "the segment property".