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Sublinear Elliptic Equation on Fractal Domains

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Abstract. This paper investigates sub-linear elliptic equations on self-similar fractal sets. With an appropriately defined Laplacian, we obtain the existence of nontrivial solutions of sub-linear elliptic equations

$$-\Delta u=\lambda u-a(x)|u|^{q-1}u-f(x,u),$$

with zero boundary Dirichlet conditions. The results are obtained by using Mountain Pass Lemma and Saddle Point Theorem.

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Key Words: Self-similar fractal; saddle point theorem; elliptic equation; mountain pass lemma; Laplacian operator.

1 Introduction

We consider the sub-linear equations

$$\begin{cases} -\Delta u=\lambda u-a(x)|u|^{q-1}u-f(x,u), & x\in K\setminus V_0, \\ u=0, & x\in V_0, \end{cases} \tag{1.1}$$

where K is a self-similar fractal domain in $R^{N-1}(N\geq 2)$ and V_0 is its boundary. Δ is the Laplacian operator on the fractal domain K. The coefficient λ is a real parameter. q is a constant $(0<q<1)$ and $a(x)$ is a bounded function on K. The function $f:K\times R\to R$ is continuous and satisfies some growth restrictions both near zero and at infinity. There has been an extensive study of problem (1.1) on classical domains, that is, K is an open set of R^n.

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Here we study (1.1) on a class of self-similar fractal domains. Since there is no concept of a generalized derivative of a function, we need to clarify the notion of differential operators such as the Laplacian on fractal domains. By [1–3] we can define Laplacian on some self-similar fractals. Once a Laplacian is defined, we may construct a Hilbert space and then establish compactness theorems allowing (1.1) to be studied. We get the existence of multiple non-trivial solutions of (1.1) on K in this paper. In Section 2 we review the definition of Laplacian on K given by Kigami [1, 2]. We introduce an energy form on K which leads to a Hilbert space $H_0^1(K)$ of functions of finite energy and we also define Green's operator on K which is the inverse of $-\Delta$ in $H_0^1(K)$. Due to the geometry of self-similar fractal domains, there is a Sobolev-like inequality on them, see [4]. We obtain some compactness results by using this inequality. In Section 3 we define a functional I associated with the equation (1.1) and give our main results. In Section 4, we state some lemmas and their proofs. The proofs of our main results are presented in Section 5 by using the Mountain Pass Lemma and the Saddle Point Theory.

In [3,5], some existence results are obtained for sup-linear elliptic equations on a class of fractal domains. In [6] we studied asymptotically linear elliptic equations and got the existence of nontrivial non-negative solution. In this paper, we will consider sub-linear elliptic problem and establish some existence results.

2 Preliminaries

Definition 2.1. *Let $(r_1,r_2,\cdots,r_N)$ be a vector with each $r_i\in(0,1)$. The numbers r_i may be interpreted as contraction factors for the mappings F_i, that is,*

$$|F_i(x)-F_i(y)|\le r_i|x-y|. \tag{2.1}$$

Let K be a connected self-similar invariant set for the iterated function system of contractive similarities F_i on some Euclidean space R^n, namely,

$$K=\bigcup_{i=1}^{N}F_iK. \tag{2.2}$$

We call K defined as above a self-similar fractal domain.

The boundary of K consists of the fixed points q_j of the mappings F_i, we call it V_0. We assume

$$F_iK\bigcap F_jK\subseteq F_iV_0\bigcap F_jV_0, \qquad \text{for } i\neq j, \tag{2.3}$$

so the cells F_iK intersect at images of boundary points only.

Let $\omega=(\omega_1,\omega_2,\cdots,\omega_m)$ denote an n-multiple index with each $\omega_k\in\{1,2,\cdots,N\}$. As in [1, 2] we create collections W_n of r_ω with the order of $(r_{max})^n$, where r_{max} and r_{min} denote the maximum and minimum values of r_i, and $r_\omega=r_{\omega_1}\cdot r_{\omega_2}\cdots r_{\omega_m}$. We assume

$$r_{min}\cdot(r_{max})^n\le r_\omega\le(r_{max})^n. \tag{2.4}$$