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Global Weak Solutions for the Weakly Dissipative Camassa-Holm Equation

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Abstract. In this paper we prove the existence and uniqueness of global weak solutions to the weakly dissipative Camassa-Holm equation.

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1 Introduction

The Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \qquad t>0, \ x\in\mathbb{R},$$

is a model for wave motion on shallow water, where $u(t,x)$ represents the fluid's free surface above a flat bottom (or equivalently, the fluid velocity at time $t\geq 0$ in the spatial x direction).

Since the equation was derived physically by Camassa and Holm [1, 2], many researchers have paid extensive attention to it. The equation has a bi-Hamiltonian structure [3] and is completely integrable [2, 4–6]. It is a re-expression of geodesic flow on the diffeomorphism group of the circle [7] and geodesic exponential maps of the Virasoro group [8]. Its solitary waves are peaked [5, 9], and they are orbitally stable and interact like solitons [9–11]. These peaked waves are analogous to the exact traveling wave solutions of the governing equations for water waves representing waves of great height- see the recent discussions in [12, 13].

The Cauchy problem of the Camassa-Holm equation has been studied extensively. It has been shown that this equation is locally well-posed [14–19] for initial data $u_0 \in$

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$H^s(\mathbb{R})$ with $s>3/2$. More interestingly, it has not only global strong solutions modelling permanent waves [14, 18–21] and but also blow-up solutions modelling wave breaking [14–21]. On the other hand, it has global weak solutions with initial data $u_0\in H^1(\mathbb{R})$, cf. [22–24]. Morover, the initial boundary value problem for the Camassa-Holm equation on the half line and on a finite interval were studied recently in [25, 26]. The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solitons and models wave breaking [2, 14, 15].

In general, it is difficult to avoid energy dissipation mechanisms in a real world. Ott and Sudan [27] investigated how the KdV equation was modified by the presence of dissipation and the effect of such dissipation on the solitary solution of the KdV equation, and Ghidaglia [28] investigated the long time behavior of solutions to the weakly dissipative KdV equation as a finite dimensional dynamical system.

Similarly, we would like to consider the dissipative Camassa-Holm equation:

$$u_t-u_{txx}+3uu_x+L(u)=2u_xu_{xx}+uu_{xxx},\quad t>0, x\in\mathbb{R},$$

where $L(u)$ is a dissipative term, L can be a differential operator or a quasi-differential operator according to different physical situations. We are interested in the effect of the weakly dissipative term on the Camassa-Holm equation. In the paper, we would like to consider the Cauchy problem of the weakly dissipative Camassa-Holm equation:

$$\begin{cases} u_t-u_{txx}+3uu_x+\lambda(u-u_{xx})=2u_xu_{xx}+uu_{xxx}, & t>0, x\in\mathbb{R},\\ u(0,x)=u_0(x), \quad x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where $L(u)=\lambda(I-\partial_x^2)u$ is the weakly dissipative term and $\lambda>0$ is a constant.

The local well-posedness, global existence and blow-up phenomena of the Cauchy problem of Eq. (1.1) on the line [29] and on the circle [30] were studied recently. We found that the behaviors of Eq. (1.1) are similar to the Camassa-Holm equation in a finite interval of time, such as, the local well-posedness and the blow-up phenomena, and that there are considerable differences between Eq. (1.1) and the Camassa-Holm equation in their long time behaviors. The global solutions of Eq. (1.1) decay to zero as time goes to infinite. This long time behavior is an important feature that the Camassa-Holm equation does not possess.

Eq. (1.1) has the same blow-up rate as the Camassa-Holm equation does when the blow-up occurs, cf. [29, 30]. This fact shows that the blow-up rate of the Camassa-Holm equation is not affected by the weakly dissipative term. But the occurrence of blow-up of Eq. (1.1) is affected by the dissipative parameter, cf. [29, 30].

With $y=u-u_{xx}$, Eq. (1.1) takes the form:

$$\begin{cases} y_t+uy_x+2u_xy+\lambda y=0, & t>0, x\in\mathbb{R},\\ y(0,x)=u_0-u_{0,xx}, & x\in\mathbb{R}. \end{cases} \tag{1.2}$$