JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **24**, No. 2, pp. 114-139

doi: 10.4208/jpde.v24.n2.2
May 2011

Construction of Green's Functions for the Two-Dimensional Static Klein-Gordon Equation

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Received 29 December 2009; Accepted 10 March 2011

Abstract. In contrast to the cognate Laplace equation, for which a vast number of Green's functions is available, the field is not that developed for the static Klein-Gordon equation. The latter represents, nonetheless, a natural area for application of some of the methods that are proven productive for the Laplace equation. The perspective looks especially attractive for the methods of images and eigenfunction expansion. This study is based on our experience recently gained on the construction of Green's functions for elliptic partial differential equations. An extensive list of boundary-value problems formulated for the static Klein-Gordon equation is considered. Computer-friendly representations of their Green's functions are obtained, most of which have never been published before.

AMS Subject Classifications: 35J08, 65N80

Chinese Library Classifications: O175.2, O241

Key Words: Static Klein-Gordon equation; Green's function.

1 Introduction

To prevent a possible confusion as to the subject of the present study, note that our focus is not on the hyperbolic Klein-Gordon equation representing a classical mathematical model in quantum field theory and for which the Green's function formalism is well developed. This project deals instead with the elliptic two-dimensional static Klein-Gordon equation (SKGE)

$$\nabla^2 u(P) - k^2 u(P) = 0 \tag{1.1}$$

with ∇^2 representing the Laplace operator written in the coordinates of point P and the parameter k is a real constant.

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To our best knowledge, [1] and [2] represent the only book-format publications covering, to a certain extend, the Green's function topic for the SKGE. Just a limited number of boundary-value problems has been reviewed in those books, with the method of eigenfunction expansion being used. The present study aims at obtaining computer-friendly forms of Green's functions for a significant list of problems stated for the SKGE.

The Green's function $G(P,Q)$ for the SKGE will be introduced, in this study, by setting up the homogeneous boundary-value problem

$$M_i u(P) \equiv \alpha_i(P)\frac{\partial u(P)}{\partial n_i} + \beta_i(P)u(P) = 0, \qquad P \in \Gamma_i, \tag{1.2}$$

for the nonhomogeneous equation

$$\nabla^2 u(P) - k^2 u(P) = -f(P), \qquad P \in \Omega, \tag{1.3}$$

where: Ω represents a simply connected region in two-dimensional Euclidean space with $\Gamma = \bigcup_{i=1}^{m} \Gamma_i$ denoting a piecewise smooth contour of Ω; $\alpha_i(P)$ and $\beta_i(P)$ represent given functions defined on Γ in such a way that at least one of them is nonzero for every piece Γ_i of Γ; and n_i represents the normal direction to Γ_i at the point P. The right-hand side function $f(P)$ in (1.3) is supposed to be integrable on Ω.

Assume that the boundary-value problem in (1.2) and (1.3) is well-posed. This implies, in fact, that it has a unique solution or, in other words, the corresponding homogeneous problem, with $f(P) \equiv 0$, has only the trivial $u(P) \equiv 0$ solution. With this, we define the Green's function for the SKGE. That is, if, for any integrable on Ω right-hand side term $f(P)$ in (1.3), the solution to the boundary-value problem in (1.2) and (1.3) is found in the form

$$u(P) = \int\int_{\Omega} G(P,Q) f(Q) \mathrm{d}\Omega(Q), \tag{1.4}$$

then the kernel $G(P,Q)$ of the above representation is said to be the *Green's function* for the homogeneous problem corresponding to that of (1.2) and (1.3).

A standard terminology will apply, according to which P and Q in (1.4) are referred to as the field (observation) point and the source point, respectively.

For any location of the source point $Q \in \Omega$, the Green's function, as a function of the coordinates of the observation point P, holds the following properties (being referred to herein as the *defining properties*):

1. at any point $P \in \Omega$, except at $P = Q, G(P,Q)$ satisfies the homogeneous equation in (1.1), that is
$$(\nabla^2 - k^2)G(P,Q) = 0, \quad P \neq Q.$$
2. For $P \to Q$, $G(P,Q)$ approaches infinity like the modified cylindrical Bessel (or Macdonald) function $K_0(k|P-Q|)$ of the second kind of order zero.
3. $G(P,Q)$ satisfies the boundary conditions in (1.2), that is
$$M_i G(P,Q) = 0, \quad P \in \Gamma_i, \ i = \overline{1,m}.$$