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Nonradial Entire Large Solutions of Semilinear Elliptic Equations

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Abstract. We consider the problem of whether the equation $\Delta u = p(x)f(u)$ on $\mathbf{R}^N$, $N \geq 3$, has a positive solution for which $\lim_{|x|\to\infty} u(x) = \infty$ where f is locally Lipschitz continuous, positive, and nondecreasing on $(0,\infty)$ and satisfies $\int_1^\infty [F(t)]^{-1/2}\mathrm{d}t = \infty$ where $F(t) = \int_0^t f(s)\mathrm{d}s$. The nonnegative function p is assumed to be asymptotically radial in a certain sense. We show that a sufficient condition to ensure such a solution u exists is that p satisfies $\int_0^\infty r \min_{|x|=r} p(x)\mathrm{d}r = \infty$. Conversely, we show that a necessary condition for the solution to exist is that p satisfies $\int_0^\infty r^{1+\varepsilon} \min_{|x|=r} p(x)\mathrm{d}r = \infty$ for all $\varepsilon > 0$.

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1 Introduction

We consider the problem

$$\Delta u = p(x)f(u), \qquad x \in \mathbf{R}^N (N \geq 3), \tag{1.1}$$

$$u(x) \to \infty \qquad \text{as } |x| \to \infty, \tag{1.2}$$

where the nonnegative function p is locally Hölder continuous on $\mathbf{R}^N$ and the nondecreasing function f is locally Lipschitz continuous on $(0,\infty)$, satisfies $f(0)=0$, $f(s)>0$ for $s>0$, and

$$\int_1^\infty [F(t)]^{-1/2}\mathrm{d}t = \infty, \qquad F(t) \equiv \int_0^t f(s)\mathrm{d}s. \tag{1.3}$$

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A positive solution of (1.1) which satisfies (1.2) is called an entire large solution of (1.1). Our interest is in giving necessary and sufficient conditions for the existence of such a solution. In doing this, we also extend the existence results of [1–5] to a broader class of functions f.

In [1] Wood and I proved that, for $f(s)=s^\gamma$, $0<\gamma\le 1$ and p radial (i.e., $p(x)=p(|x|)$), a necessary and sufficient condition for (1.1) to have an entire large solution is that p satisfies

$$\int_0^\infty rp(r)\mathrm{d}r=\infty.$$

We then asked whether a similar result is true if p is nonradial (Remark 1 of [1]); i.e., we asked whether (1.1) will have a positive entire large solution if p is nonradial and satisfies

$$\int_0^\infty rp_*(r)\mathrm{d}r=\infty, \qquad p_*(r)\equiv\min_{|x|=r}p(x). \tag{1.4}$$

In [2] I showed that it is true if p is appropriately asymptotically radial. In particular, I showed that if f is sublinear (i.e., $\sup_{s\ge 1}f(s)/s\equiv\Lambda<\infty$) and the difference

$$p_{osc}(r)\equiv p^*(r)-p_*(r), \qquad \text{with } p^*(r)\equiv\max_{|x|=r}p(x)$$

satisfies

$$\int_0^\infty tp_{osc}(t)\exp\left(\frac{\Lambda}{N-2}\int_0^t sp_*(s)\mathrm{d}s\right)\mathrm{d}t<\infty, \tag{1.5}$$

then (1.4) is both necessary and sufficient for (1.1) to have an entire large solution. Yang [3] extended this to functions f satisfying

$$\int_1^\infty \frac{\mathrm{d}s}{f(s)}=\infty, \tag{1.6}$$

provided p satisfies

$$\int_0^\infty rp_{osc}(r)f\circ G^{-1}\left(\frac{2}{N-2}\int_0^r sp^*(s)\mathrm{d}s\right)\mathrm{d}r<\infty, \tag{1.7}$$

where G^{-1} is the inverse of the function

$$G(r)=\int_1^r \frac{\mathrm{d}s}{f(s)}.$$

We note that condition (1.6) includes the sublinear case in [2], but the condition (1.7) on p is, in general, more restrictive than (1.5) since, for example, if f is linear it requires p_{osc} to decay faster than an exponential in p^* rather than p_*. El Mabrouk and Hansen [4] showed that the condition (1.5) could be weakened considerably to

$$\int_0^\infty tp_{osc}(t)\left(1+\int_0^t sp_*(s)\mathrm{d}s\right)^{\gamma/(1-\gamma)}\mathrm{d}t<\infty,$$