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Solitary Water Waves for a 2D Boussinesq Type System

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Abstract. We prove the existence of solitons (finite energy solitary wave) for a Boussinesq system that arise in the study of the evolution of small amplitude long water waves including surface tension. This Boussinesq system reduces to the generalized Benney-Luke equation and to the generalized Kadomtsev-Petviashivili equation in appropriate limits. The existence of solitons follows by a variational approach involving the Mountain Pass Theorem without the Palais-Smale condition. For surface tension sufficiently strong, we show that a suitable renormalized family of solitons of this model converges to a nontrivial soliton for the generalized KP-I equation.

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1 Introduction

The study of the evolution of surface water waves has brought the attention to many mathematicians and physics after the observations of Scott Russell in the nineteenth century. As it is well known, one of the major difficulty in the study of three dimensional water waves is that the full water wave problem is described in terms of two unknowns: the velocity potential and the free surface elevation. By eliminating the vertical variable from the equations by an approximation process, the study of water waves is reduced to determine the free surface elevation and the velocity potential on the free surface. In particular, Models for dispersive and weakly nonlinear long water waves with small amplitude in finite depth can be described through approximation of the full water wave problem, by imposing some restrictions on the parameters that affect the propagation of gravity water waves, as the nonlinearity (amplitude parameter) and the dispersion

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(long-wave parameter), and also by assuming that the free surface elevation, its derivatives, and the derivatives of the velocity potential are small quantities compared with the amplitude parameter and the long wave parameter.

Among the most recent papers for the formulation of water waves related with the (KP) equation, the Benney-Luke equation, and Boussinesq systems, we have to mention the work by Ablowitz, Fokas, and Musslimani in [1]. In the particular case of the Benney-Luke models, Pego and Quintero in [2] (see also the work by Quintero in [3]) obtained a model in the presence of surface tension, Benney and Luke in [4] derived a similar equation assuming that the amplitude parameter and the long wave parameter were equal, in the absence of surface tension. We also know that Milewski and Keller in [5] considered a related work without the long-wave assumption. In those works, the main step in deriving the Benney-Luke models consists in eliminating the free surface elevation η from the system describing the full water wave problem.

We will see in this work that the study of the evolution of long water waves with small amplitude in the presence of surface tension is reduced to studying solutions $(\Phi,\eta)(x,y,t)$ of Boussinesq type system

$$\begin{cases} \eta_t + \epsilon\nabla\cdot\left(\eta\left(\Phi_x^p,\Phi_y^p\right)\right) + \Delta\Phi - \dfrac{\mu}{6}\Delta^2\Phi = 0, \\ \Phi_t + \eta - \mu(\sigma-1/2)\Delta\eta + \dfrac{\epsilon}{p+1}\left(\Phi_x^{p+1} + \Phi_y^{p+1}\right) = 0, \end{cases} \tag{BTSp}$$

where ϵ is the amplitude parameter (nonlinearity coefficient), $\mu = (h_0/L)^2$ is the long-wave parameter (dispersion coefficient), σ is the Bond number (associated with the surface tension), and $p=1$. The variable Φ is the rescale nondimensional velocity potential on the bottom $z=0$, and the variable η is the rescaled free surface elevation.

From the physical view point, we believe that the Boussinesq type system described above could be considered as much more realist than the Benney-Luke models to study long water wave of small amplitude. The main reason to assert this is that the derivation of Benney-Luke model required the elimination the surface elevation η up to order two in ϵ and μ, to obtain a single equation in the variable Φ (essentially the rescaled velocity potential ϕ at the bottom $z=0$).

One of the main features that makes the system (BTSp) very interesting from the physical and numerical view points is that well known water wave models as the generalized Benney-Luke equation and the generalized Kadomtsev-Petviashivili equation emerge from this Boussinesq type system (BTSp) (up to some order with respect to ϵ and μ), giving some sort of physical sense to this system.

From the mathematical view point, it is important to establish the existence of solutions for the system (BTSp) for fixed values of the parameters ϵ and μ, but from the physical relevance is very important to show that appropriate order-one solutions exist for arbitrarily small value of the parameters. For instance, if we balance the effect of nonlinearity and dispersion assuming that $\mu=\epsilon^{2/p+1}$, and seek a travelling-wave solution of