JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **23**, No. 2, pp. 203-208

doi: 10.4208/jpde.v23.n2.6
May 2010

SHORT COMMUNICATION SECTION

Some Geometric Flows on Kähler Manifolds

SUN Xiaowei and WANG Youde*

Institute of Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100190, China.

Received 28 October 2009; Accepted 6 January 2010

Abstract. We define a kind of KdV (Korteweg-de Vries) geometric flow for maps from a real line or a circle into a Kähler manifold (N,J,h) with complex structure J and metric h as the generalization of the vortex filament dynamics from a real line or a circle. By using the geometric analysis, the existence of the Cauchy problems of the KdV geometric flows will be investigated in this note.

AMS Subject Classifications: 35Q53, 35Q35, 53C44

Chinese Library Classifications: O175.29

Key Words: KdV geometric flow; conservation law; local and global existence.

1 The definition of some geometric flow

Let (N,J,h) be a Kähler manifold with complex structure J and metric h. For any smooth map $u(x,t)$ from $S^1\times\mathbb{R}$ into (N,J,h), let ∇_x denote the covariant derivative $\nabla_{\frac{\partial}{\partial x}}$ on the pull-back bundle $u^{-1}TN$ induced from the Levi-Civita connection ∇ on N. For the sake of convenience, we always denote $\nabla_x u$ and $\nabla_t u$ by u_x and u_t respectively. The energy of a smooth map $v:S^1\to N$ is defined as

$$E_1(v)\equiv\frac{1}{2}\int_{S^1}|v_x|^2\mathrm{d}x,$$

and the tension field of v is written by $\tau(v)\equiv\nabla_x v_x$.

For the maps from a unit circle S^1 or a real line $\mathbb{R}$ into N, we define a class of geometric flows, which we would like to call **KdV geometric flow** (Korteweg-de Vries), as follows:

$$\frac{\partial u}{\partial t}=\nabla_x^2 u_x+\frac{1}{2}R(u_x,J_u u_x)J_u u_x, \tag{1.1}$$

*Corresponding author. *Email address:* `wyd@math.ac.cn` (Y. Wang)

where R is the curvature tensor on N and $J_u \equiv J(u)$. If (N,J,h) is a locally Hermitian symmetric space, it is easy to verify that the geometric flow is an energy conserved system. Moreover, the flow on any Kähler manifold N always preserves the following moment density

$$E_2(u) \equiv \int \langle \nabla_x u_x, J u_x \rangle \mathrm{d}x.$$

Let (Σ,h) be a Riemann surface. It is easy to verify that on Σ there holds true

$$R(u_x, J_u u_x) J_u u_x = K(u)|u_x|^2 u_x,$$

where $K(\cdot)$ is the Gauss curvature (sectional curvature) function on (Σ,h). So, the KdV geometric flow from S^1 or $\mathbb{R}$ into (Σ,h) can be written by

$$\frac{\partial u}{\partial t} = \nabla_x^2 u_x + \frac{1}{2} K(u)|u_x|^2 u_x. \tag{1.2}$$

It is not difficult to see that in the case Σ is a 2-dimensional sphere S^2 the above KdV flow can be derived from the following curve flow $\mathbf{u}$ from $S^1 \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$ into Euclidean space $\mathbb{R}^3$

$$\mathbf{u}_t = \mathbf{u}_s \times \mathbf{u}_{ss} + \beta\Big(\mathbf{u}_{sss} + \frac{3}{2}\mathbf{u}_{ss} \times (\mathbf{u}_s \times \mathbf{u}_{ss})\Big), \tag{1.3}$$

where $\times$ denotes the cross product in $\mathbb{R}^3$. For the mechanical background of the curve flow we refer to [2–4].

Now we recall another geometric flow, namely Schrödinger flow, from $S^1 \times \mathbb{R}$ or $\mathbb{R} \times \mathbb{R}$ into a Kähler manifold (N,J,h) formulated by (see, e.g., [5–8])

$$\frac{\partial u}{\partial t} = J(u) \nabla_x u_x = J(u)\tau(u), \tag{1.4}$$

which is a Hamilton system with the energy functional.

Here we should mention that Terng and Uhlenbeck [9] constructed via gauge transformations an isomorphism from the phase space of Schrödinger flow from $\mathbb{R}$ into a Grassmann manifold to the phase space of the matrix valued Schrödinger equation so that the Schrödinger flow corresponds to the matrix valued Schrödinger flow. In fact, The theory of Terng and Uhlenbeck also implies that the above KdV geometric flow from $\mathbb{R}$ into a Grassmann manifold corresponds to the matrix valued KdV flow associated Grassmannian symmetric Lie algebra. Matrix valued Schrödinger equations and KdV flows (or vector valued mKdV equation) associated Hermitian symmetric spaces (Lie algebra) was introduced in [4] and [10].

We define another geometric flow for maps from S^1 or $\mathbb{R}$ into a Kähler manifold (N,J,h) as follows

$$u_t = \alpha J_u \tau(u) + \beta\left(\nabla_x^2 u_x + \frac{1}{2} R(u_x, J_u u_x) J_u u_x\right), \tag{1.5}$$