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On the Decay Property of Solutions of Viscous Burgers Equation with Boundary Feedback

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Abstract. In this article, we will investigate the viscous Burgers equation with boundary feedback. The existence of the solution is proved by constructing a convergence sequence inductively. Moreover, the decay property of the solution is shown based on the maximum principle for nonlinear parabolic equations.

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1 Introduction

In this paper, we consider the following system associated with the one-dimensional viscous Burgers equation with nonlocal boundary condition

$$y_t - \nu y_{xx} + y y_x = 0, \qquad 0 \le x \le 1, \ \ t \ge 0, \tag{1.1a}$$

$$y(0,t) = 0, \ \ y(1,t) = \alpha \int_0^1 y(x,t)\mathrm{d}x, \qquad t \ge 0, \tag{1.1b}$$

$$y(x,0) = y_0(x), \qquad 0 \le x \le 1, \tag{1.1c}$$

where the positive constant ν is the viscosity coefficient. In (1.1), the boundary condition at the left end $x=0$ is homogenous and at the right end $x=1$ is nonlocal, i.e., with an integral term. In this article, the factor α is assumed to be a constant satisfying

$$0 < |\alpha| < 1. \tag{1.2}$$

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Moreover, the initial datum y_0 is supposed to be in $C(0,1)$ satisfying the compatibility conditions

$$y_0(0)=0, \quad y_0(1)=\alpha\int_0^1 y_0(x)\mathrm{d}x, \tag{1.3}$$

for instance, $y_0(x)=\sin 2\pi x$.

Burgers equation was developed by Burgers as a simple fluid flow model which exhibits some of the important aspects of turbulence, and was later derived by Lighthill as a second-order approximation to the one-dimensional unsteady Navier-Stokes equations. Now Burgers equation is viewed as a one-dimensional simple model for convection-diffusion phenomena such as shock waves, supersonic flow about airfoils, traffic flows, acoustic transmission, etc. When setting $\nu=0$ in (1.1a), we then get the so-called nonviscous Burgers equation

$$y_t+\left(\frac{y^2}{2}\right)_x=0.$$

The nonviscous Burgers equation is often considered as the analogue of the equation of Euler for the incompressible inviscid fluids in one-dimension. Hopf in [1] showed that solutions of viscous Burgers equation can converge to shock solutions as the viscosity $\nu\to 0$.

In the past decades, there has been considerable attention to the control problems of Burgers equation. The study of control problems for the Burgers equation is a natural first step developing methods for control of flows. To begin with, let us recall some previous results on this issue for the nonviscous and viscous Burgers equations. For the controllability problem, we refer the readers to [2–9] among others. As far as the stabilization problem for the viscous Burgers equation is concerned, we refer the readers to [10–15]. It is well known that the solution of the viscous Burgers equation with the homogeneous Dirichlet boundary condition on the finite interval $[0,\ell]$ decays exponentially in the H_0^1-norm in the absence of control, see, e.g., [10, 15]. However in the open-loop case, the rate at which the solution goes to zero in the space $H_0^1(0,\ell)$ depends on the viscosity ν , which is small for small viscosity. To enhance the stability of the solution by requiring a certain fixed exponential decay rate, feedback control laws were designed in [10] and [11], where the linear quadratic regulator (LQR) theory was applied. In [10], the feedback control law was obtained from the linear regulator problem with bounded input and bounded output operators. While in [11] the feedback control law was obtained from the linear regulator problem with bounded input and unbounded output operators. In both papers, the feedback control law is of the form

$$Ky=\int_0^\ell k(s)y(s)\mathrm{d}s, \tag{1.4}$$

where the feedback functional gain $k(\cdot)\in L^2(0,\ell)$. In [11], such kind of feedback law was also applied to the boundary control problem for Burgers equation to yield a closed-loop