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Traveling Waves and Capillarity Driven Spreading of Shear-Thinning Fluids

Joseph A. Iaia*

P.O. Box 311430, Department of Mathematics, University of North Texas, Denton, TX 76203-1430, USA.

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Abstract. We study capillary spreadings of thin films of liquids of power-law rheology. These satisfy

$$u_t+(u^{\lambda+2}|u_{xxx}|^{\lambda-1}u_{xxx})_x=0,$$

where $u(x,t)$ represents the thickness of the one-dimensional liquid and $\lambda>1$. We look for traveling wave solutions so that $u(x,t)=g(x+ct)$ and thus g satisfies

$$g'''=\frac{|g-\epsilon|^{\frac{1}{\lambda}}}{g^{1+\frac{2}{\lambda}}}\operatorname{sgn}(g-\epsilon).$$

We show that for each $\epsilon>0$ there is an infinitely oscillating solution, g_ϵ, such that

$$\lim_{t\to\infty} g_\epsilon=\epsilon$$

and that $g_\epsilon\to g_0$ as $\epsilon\to 0$, where $g_0\equiv 0$ for $t\geq 0$ and

$$g_0=c_\lambda|t|^{\frac{3\lambda}{2\lambda+1}} \quad \text{for} \quad t<0$$

for some constant c_λ.

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1 Introduction

In this work, we study *capillary spreadings* of thin films of liquids of power-law rheology, also known as Ostwald-de Waele fluids. The following equation for one-dimensional

*Corresponding author. *Email address:* `iaia@unt.edu` (J. A. Iaia)

motion was derived in [1, 2] and is

$$u_t+\left(u^{\lambda+2}|u_{xxx}|^{\lambda-1}u_{xxx}\right)_x=0,$$

where λ is a real constant and $u(x,t)$ represents the thickness of the one-dimensional liquid film at position x and time t. See also [3, 4]. When $\lambda>1$, the fluid is called *shear thinning* and the viscosity tends to zero at high strain rates [5]. Typical values for λ are between 1.7 and 6.7 [6].

For *gravity driven* spreadings studied in [7], $u(x,t)$ satisfies

$$u_t-\left(u^{\lambda+2}|u_x|^{\lambda-1}u_x\right)_x=0.$$

If we look for traveling wave solutions of the above equation so that $u(x,t)=g(x+ct)$ for some nonzero $c\in\mathbf{R}$, we obtain

$$cg'=\left(g^{\lambda+2}|g'|^{\lambda-1}g'\right)'$$

and thus

$$c(g-K)=g^{\lambda+2}|g'|^{\lambda-1}g'$$

for some constant K. In the case $K=0$ we obtain

$$g(z)=d(z-z_0)^{\frac{\lambda}{2\lambda+1}}$$

for some constant d which represents a current advancing with constant speed, c, and front located at $x=-ct-z_0$. In particular, this differential equation has no oscillatory traveling wave solutions. Similarly, in the case $K\neq 0$ there are no oscillatory traveling wave solutions. If $g'(m_1)=g'(m_2)=0$ with $m_1<m_2$, then it follows from the differential equation that $g(m_1)=K=g(m_2)$. Now let M be the maximum (or minimum) of g on $[m_1,m_2]$. Then $g'(M)=0$ and thus $g(M)=K$. Thus $g\equiv K$ on $[m_1,m_2]$.

In this paper, we will study traveling wave solutions for *capillarity-driven* spreadings in which case we obtain

$$cg'+\left(g^{\lambda+2}|g'''|^{\lambda-1}g'''\right)'=0$$

and so

$$cg+g^{\lambda+2}|g'''|^{\lambda-1}g'''=K.$$

If we expect that g will be essentially constant as $t\to\infty$, say $\epsilon>0$, then this gives the equation

$$c(g-\epsilon)+g^{\lambda+2}|g'''|^{\lambda-1}g'''=0.$$

This reduces to

$$g'''=d\,\frac{|g-\epsilon|^{\frac{1}{\lambda}}}{g^{1+\frac{2}{\lambda}}}\operatorname{sgn}(g-\epsilon),\quad \text{where } d=-\frac{c}{|c|^{1-\frac{1}{\lambda}}}.$$