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A Fourth-order Degenerate Parabolic Equation with Variable Exponent

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Abstract. In this paper we establish the existence, uniqueness and long-time behavior of weak solutions for the initial-boundary value problem of a fourth-order degenerate parabolic equation with variable exponent of nonlinearity.

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1 Introduction

Suppose that Ω is a bounded open domain of $\mathbb{R}^N$ with a $C^{1,1}$ boundary $\partial\Omega$, T is a given positive number. Denote the cylinder $Q\equiv\Omega\times(0,T]$, the lateral surface $\Gamma\equiv\partial\Omega\times(0,T]$. In this paper we consider the following fourth-order parabolic initial-boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}+\operatorname{div}\left(|\nabla\Delta u|^{p(x)-2}\nabla\Delta u\right)=0 & \text{in } Q,\\ u=\Delta u=0 & \text{on } \Gamma,\\ u(x,0)=u_0(x) & \text{on } \Omega, \end{cases} \tag{1.1}$$

where $p:\overline{\Omega}\to(1,+\infty)$ is a continuous function (called the variable exponent).

When $p>1$ is a constant, King in [1] firstly derived the equation which is relevant to capillary driven flows of thin films of power-law fluids, where $u(x,t)$ denotes the height from the surface of the oil to the surface of the solid. The exponent p is related to the rheological properties of the liquid: $p=2$ corresponds to a Newtonian liquid, whereas

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$p \neq 2$ emerges when considering "power-law" liquids. When $p > 2$ the liquid is said to be shear-thinning. Some problems related to (1.1) with constant exponent have been studied in many papers. We refer to [1–5] for details.

The study of differential equations and variational problems with nonstandard growth conditions arouses much interest with the development of elastic mechanics, electro-rheological fluid dynamics and image processing, etc. We refer the readers to [6–9] and references therein. $p(x)$-growth conditions can be regarded as a very important class of nonstandard (p,q)-growth conditions. There are already numerous results for such kind of problems (see [10–16]). The functional spaces to deal with these problems are the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and the generalized Lebesgue-Sobolev spaces $W^{k,p(x)}(\Omega)$.

In [5], Xu and Zhou have established the existence, uniqueness and some regularity results of weak solutions for problem (1.1) when $p(x)$ is a constant. This paper is a further step to extend the constant p to the variable exponent $p(x)$. As far as we know, there are no papers concerned with the fourth-order nonlinear parabolic equations involving multiple anisotropic exponents. It is not a trivial generalization of similar problems studied in the constant case. The main difficulties for treating the problem are caused by the complicated nonlinearities (it is nonhomogeneous) of problem (1.1) and the lack of a maximum principle for fourth-order equations.

The starting point of our arguments is an energy type estimate (1.6). Under the appropriate definition of weak solutions, we first combine the difference and variation techniques to construct an approximation solution sequence for problem (1.1) and establish some *a priori* estimates. Next, we draw a subsequence to obtain a limit function, and then prove this function is the unique weak solution.

Now we recall the definitions and some basic properties of the generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and the generalized Lebesgue-Sobolev spaces $W^{k,p(x)}(\Omega)$.

Set

$$C_+(\overline{\Omega}) = \Big\{ h \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} h(x) > 1 \Big\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \quad \text{and} \quad h^- = \inf_{x\in\Omega} h(x).$$

For any $p \in C_+(\overline{\Omega})$, we introduce the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions such that

$$\int_\Omega |u(x)|^{p(x)} \,\mathrm{d}x < \infty,$$

endowed with the *Luxemburg norm*

$$|u|_{p(\cdot)} = \inf\Big\{ \lambda > 0 : \int_\Omega \Big| \frac{u(x)}{\lambda} \Big|^{p(x)} \mathrm{d}x \le 1 \Big\},$$