JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
Vol. **22**, No. 2, pp. 127-152

J. Partial Diff. Eq.
May 2009

Existence and Uniqueness of BV Solutions for a Class of Degenerate Boltzmann Equations with Measures as Initial Conditions

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Received 15 May 2008; Accepted 9 January 2009

Abstract. The existence and uniqueness of the solutions for the Boltzmann equations with measures as initial value are still an open problem which is posed by P. L. Lions (2000). The aim of this paper is to discuss the Cauchy problem of the system of discrete Boltzmann equations of the form

$$\partial_t f_i + (f_i^{m_i})_x = Q_i(f_1, f_2, \cdots, f_n), \quad (m_i > 1,\ i = 1, \cdots, n)$$

with non-negative finite Radon measures as initial conditions. In particular, the existence and uniqueness of BV solutions for the above problem are obtained.

AMS Subject Classifications: 35L80, 35L60, 34A12, 74G20, 35B40

Chinese Library Classifications: O175.2

Key Words: System of discrete Boltzmann equations; BV solutions; existence and uniqueness.

1 Introduction

The existence and uniqueness of the solutions for the Boltzmann equations with measures as initial value are still an open problem which is posed by P. L. Lions, see [1]. In this paper we consider a class of degenerate Boltzmann equations which is one of the fundamental equations of kinetic theory. It is used to describe the evolutions of the free motion of non-interacting particles systems, such as rarefied gas. The equation reads as follows:

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla f = Q(f), \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \tag{1.1}$$

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where $f=f(\xi,x,t)$ is the one-particle distribution function, which has the physical meaning of mass density of particles in the (ξ,x)-space. Here variables ξ, x and t denote molecular velocity, position and time, respectively; $Q(f)$ is non-linear integral operator, called collision operator. The discretization of a velocity space in the Boltzmann equation (1.1) allows us to replace the integro-differential equation by a system of quasilinear hyperbolic equations

$$\frac{\partial f_i}{\partial t}+\xi_i\cdot\bigtriangledown f_i=Q_i(f),\qquad (x,t)\in\mathbb{R}^n\times\mathbb{R}_+, \tag{1.2}$$

where $f=(f_1,f_2,\cdots,f_n)$ is the density of particles with a velocity ξ_i, and the collision operator $Q_i(f)$ is defined by

$$Q_i(f)=\sum_{j,k}B_i^{jk}f_jf_k,\quad B_i^{jk}\neq 0\Rightarrow j\neq k,\quad B_k^{ij}=B_k^{ji}, \tag{1.3}$$

where the coefficients B_i^{jk} are assumed to be constants and have the meaning of the rates of collisions and

$$|B_i^{jk}|\leq B_*<\infty.$$

The transversality assumption (1.3) of collision terms is natural, because of conservation laws, if the pre-collision velocities are equal, then the post-collision velocities should be equal to the pre-collision velocities. Hence the collisions between particles with the same velocity will not contribute to $Q_i(f)$.

In this paper, we consider the system of discrete Boltzmann equations

$$\partial_t f_i+(f_i^{m_i})_x=Q_i(f) \tag{1.4}$$

in $G_T=\mathbb{R}\times(0,T)$ with initial conditions

$$f_i(x,0)=f_i^0(x) \tag{1.5}$$

for all $x\in\mathbb{R}$, where $f=(f_1,f_2,\cdots,f_n)$, $m_i>1\,(i=1,\cdots,n)$ and $f_i^0(x)$ is a non-negative finite Radon measure in $\mathbb{R}\equiv(-\infty,+\infty)$.

The global existence and uniqueness of mild solutions to the one-dimensional discrete-velocity Boltzmann models were first obtained by Nishida and Mimura [2] and later this small data existence theory was generalized by Tartar [3,4], and for general one-dimensional discrete Boltzmann models with self-interaction terms by Beale [5], and a one-dimensional Boltzmann-type equation with inelastic collisions by Ha [6]. In contrast, the global existence and uniqueness of mild solutions for multi-dimensional discrete-velocity Boltzmann models have been studied by Kawashima [7,8], Illner [9–11], Panferov and Heintz [12], Ha [13].

The definition of solution in the mild sense can be stated as follows.

Definition 1.1. *A vector function $f=(f_1,\cdots,f_n):G_T\mapsto(0,+\infty)$ is said to be a solution of* (1.4) *if f satisfies the following conditions:*