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Existence Results for a Class of Semilinear Elliptic Systems

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Abstract. In this paper, we study the existence of nontrivial solutions for the problem

$$\begin{cases} -\Delta u = f(x,u,v)+h_1(x) & \text{in } \Omega, \\ -\Delta v = g(x,u,v)+h_2(x) & \text{in } \Omega, \\ u=v=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is bounded domain in $\mathbb{R}^N$ and $h_1,h_2 \in L^2(\Omega)$. The existence result is obtained by using the Leray-Schauder degree under the following condition on the nonlinearities f and g:

$$\begin{cases} \lim\limits_{s,|t|\to+\infty} \dfrac{f(x,s,t)}{s} = \lim\limits_{|s|,t\to+\infty} \dfrac{g(x,s,t)}{t} = \lambda_+, & \text{uniformly on } \Omega, \\ \lim\limits_{-s,|t|\to+\infty} \dfrac{f(x,s,t)}{s} = \lim\limits_{|s|,-t\to+\infty} \dfrac{g(x,s,t)}{t} = \lambda_-, & \text{uniformly on } \Omega, \end{cases}$$

where $\lambda_+,\lambda_- \notin \{0\}\cup\sigma(-\Delta)$, $\sigma(-\Delta)$ denote the spectrum of $-\Delta$. The cases (i) where $\lambda_+ = \lambda_-$ and (ii) where $\lambda_+ \neq \lambda_-$ such that the closed interval with endpoints λ_+,λ_- contains at most one simple eigenvalue of $-\Delta$ are considered.

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1 Introduction and main results

The question of the existence of solutions of semilinear elliptic systems of the form

$$\begin{cases} -\Delta u = f(x,u,v) & \text{in } \Omega, \\ -\Delta v = g(x,u,v) & \text{in } \Omega, \end{cases} \tag{1.1}$$

subject to zero Dirichlet boundary conditions, has been the object of intensive research recently and different situations on the structure of the nonlinearities f and g are studied; see for example [1–6] and references therein.

The study of this type of problem is motivated by its various applications, for example, in reaction-diffusion equations (where the steady state equations are elliptic systems), in newtonian fluids and in the study of torsional creep (see [7]).

In this work, we are concerned with the existence of nontrivial solutions for the following class of elliptic systems

$$\begin{cases} -\Delta u = f(x,u,v) + h_1(x) & \text{in } \Omega, \\ -\Delta v = g(x,u,v) + h_2(x) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$ and $h = (h_1,h_2)$ is $\left(L^2(\Omega)\right)^2$ nonnull function. We assume that

$$f,g:\ \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

are continuous functions satisfying the condition below:

$$\begin{cases} |f(x,s,t)| \leq C_1(1+|s|+|t|), & \forall s,t \in \mathbb{R}, \text{ a.e. } x \in \Omega, \\ |g(x,s,t)| \leq C_2(1+|s|+|t|), & \forall s,t \in \mathbb{R}, \text{ a.e. } x \in \Omega, \end{cases} \tag{1.3}$$

where C_1, C_2 are positive constants. Problem (1.2) is supposed to be nonvariational, that is, the conditions defined below are not satisfied by the nonlinearities f and g.

Definition 1.1. *We say that the system* (1.2) *is variational if either one of the following conditions holds:*

1. *There is a real-valued differentiable function $F(x,u,v)$ for $(x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$ such that*

$$\frac{\partial F(x,u,v)}{\partial u} = f(x,u,v), \quad \frac{\partial F(x,u,v)}{\partial v} = g(x,u,v).$$

In this case, the system is said to be of Gradient type.

2. *There is a real-valued differentiable function $H(x,u,v)$ for $(x,u,v) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$ such that*

$$\frac{\partial H(x,u,v)}{\partial u} = g(x,u,v), \quad \frac{\partial H(x,u,v)}{\partial v} = f(x,u,v).$$

In this case, the system is said to be of Hamiltonian type.