J. Partial Diff. Eqs. **21**(2008), 377–383

Vol.21 No.4

GLOBAL EXISTENCE AND EXTINCTION FOR A DEGENERATE NONLINEAR DIFFUSION PROBLEM WITH NONLINEAR GRADIENT TERM AND ABSORPTION

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(Received Mar. 13, 2007)

Abstract Existence and extinction in finite time of global weak solutions for the problem (P) are proved.

Key Words Degenerate nonlinear diffusion equation; gradient term; global existence; extinction.

2000 MR Subject Classification 35K65, 35K55.

Chinese Library Classification O175.29.

1. Introduction

We consider the following degenerate nonlinear diffusion problem with a nonlinear gradient term and absorption

$$(P)\begin{cases} u_t = \Delta u^m - \lambda u^p + |\nabla u^\alpha|^q & \text{in } Q = \Omega \times (0,\infty) \\ u(x,t) = 0 & \text{on } S = \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $m \geq 1, p > 0, \lambda > 0, 1 \leq q < 2, \alpha > \frac{m}{2}$ and $0 \leq u_0 \in L^\infty(\Omega)$.

With regard to the equation (P) for the case $m = \alpha = 1$, several authors have studied the existence of global and nonglobal positive solutions and total extinction in the finite time under certain assumptions on p, q, λ, N and Ω. (see [1], [2] and the reference therein in [3]).

On the other hand,the existence of the solutions of the equations

$$u_t = \Delta u^m + |\nabla u^\alpha|^q \qquad \text{and} \qquad u_t = \Delta u^m + u^p - |\nabla u^\alpha|^q$$

has been studied in [4] and [5] respectively under some some assumptions on initial data, p, q, λ, N and Ω. (cf. the reference therein in [3], [6] and [7]).

The aim of this paper is to prove the existence of global weak solutions and extinction properties of solutions for a degenerate nonlinear diffusion problem with (P) under some assumptions.

This paper is structured as follows. In the next section we establish the existence of global weak solutions. In the third section we prove the total extinction in finite time.

2. Existence of Global Weak Solutions

In this paper, we use the following definition of weak solution.

Definition *Given $0 \leq u_0 \in L^\infty(\Omega)$ by a weak solution of the problem (P) on Q_T we mean a function $u \in L^\infty(\Omega \times (0,T))$ such that $u^m \in L^2(0,T;H_0^1(\Omega))$, $u^\alpha \in L^q(0,T;W_0^{1,q}(\Omega))$ and satisfies the identity*

$$\int_{Q_T} -u\varphi_t + \nabla u^m \cdot \nabla\varphi + \lambda u^p \varphi - |\nabla u^\alpha|^q \varphi = \int_\Omega u_0(x)\varphi(x,0)\mathrm{d}x$$

for any test function $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(Q_T)$ with $\varphi(T) = 0$. We shall say that u is a global weak solution of the problem (P) if u is a weak solution on Q_T for all positive T.

Let us now state our existence result.

Theorem 2.1 *Given $0 \leq u_0 \in L^\infty(\Omega)$ there exists a global weak solution of problem (P) such that*

$$\|u\|_{L^\infty(\Omega)} \leq C(\|u_0\|_{L^\infty(\Omega)})$$

In proving the existence of the solution of (P) one standard approach is to approximate the problem with a sequence of nondegenerate problems which can be solved in a classical sense. We can find the functions $u_{0n} \in C_0^\infty(\Omega)$ satisfying

$$u_{0n} \geq \frac{1}{n}, \quad \|u_{0n}\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} + 1,$$

and

$$\|u_{0n} - u_0\|_{L^1(\Omega)} \to 0 \quad \text{as } n \to \infty.$$

Consider the approximated problems

$$(P_n)\begin{cases} (u_n)_t = \Delta u_n^m - \lambda u_n^p + |\nabla u_n^\alpha|^q + \lambda(\frac{1}{n})^p & \text{in } Q_T = \Omega \times (0,T) \\ u_n = \frac{1}{n} & \text{on } S_T = \partial\Omega \times (0,T) \\ u_n(x,0) = u_{0n}(x) & \text{in } \Omega. \end{cases}$$

The existence of a smooth solution u_n to (P_n) follows from standard parabolic theory.

Proposition 2.1 *Let u_n be the solution of the problems (P_n). Then there exists a constant C depending only on $\|u_0\|_{L^\infty(\Omega)}$, such that*

$$\frac{1}{n} \leq u_n \leq C. \tag{2.1}$$

Proof By maximum principle we can get (2.1).