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BLOW-UP RATE OF SOLUTIONS FOR P-LAPLACIAN EQUATION

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Abstract In this note we consider the blow-up rate of solutions for p-Laplacian equation with nonlinear source, $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u)+u^q$, $(x,t) \in R^N \times (0,T)$, $N \geq 1$. When $q > p-1$, the blow-up rate of solutions is studied.

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1. Introduction

In this paper, we consider the Cauchy problem

$$\begin{cases} u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + u^q, & (x,t) \in R^N \times (0,T), \\ u(x,0) = u_0(x), & x \in R^N, \end{cases} \tag{1.1}$$

where $p > 2$, $q > p-1$.

The mathematical modal (1.1) is a class of degenerate diffusion equations with strongly nonlinear sources. It appears in the theory of non-Newtonian fluids and is a nonlinear form of heat equation. It has attracted many authors to study it. The main feature of this class of equations is the interplay between the degeneracy in the principal part and the growth of forcing term. The local and global existences of solution in time, uniqueness and regularity for solution of (1.1) have been investigated (see [1–3]). It is well known(see [3]) that when $p > 2$, $1 < q \leq p-1+\frac{p}{N}$, the solutions of (1.1) always blow up in finite time, while for $q > p-1+\frac{p}{N}$ the blow-up occur if u_0 is large enough. In the latter case there also exist small solutions which are global in time. By finite blow-up we mean there is a $T > 0$ such that $\|u(\cdot,t)\|_\infty$ is finite for all $t \in [0,T)$, but

$$\lim_{t\to T^-} \sup \|u(\cdot,t)\|_\infty = +\infty,$$

and T is called blowing up time. In this note, we are interested in the blow-up rate for the blow-up solution of (1.1). Here is our main result

Denote

$$B(x_0,\rho)=\{x\in R^N;\ \ |x-x_0|<\rho\}$$

Theorem 1 *Let $q>p-1$, and $T<\infty$ be the blowing up time for solution u of* (1.1). *Suppose that $u_0(x)=u_0(|x|)$ is decreasing. Then for all $t\in[0,T)$ and $\delta>0$*

$$\|u(\cdot,t)\|_\infty\le C(\delta)(T-t)^{-\frac{1}{q-1}},\quad x\in R^N\setminus B(0,\delta), \tag{1.2}$$

where $C(\delta)$ is a constant depending on δ.

Theorem 2 *Let $q>p-1$ and $u_0\in L^1_{loc}(R^N)$. Suppose that u is a non-negative supersolution of* (1.1) *and that T is the blow-up time of u. Then there exists a constant C such that for all $0<t<T,\ \ x_0\in R^N$*

$$\int_{B(x_0,1)}u(x,t)\mathrm{d}x\le C(T-t)^{-\frac{1}{q-1}}; \tag{1.3}$$

$$\int_t^T\int_{B(x_0,1)}u^q\mathrm{d}x\mathrm{d}t\le C(T-t)^{-\frac{1}{q-1}}. \tag{1.4}$$

The weak solution of (1.1) is defined in the following sense.

Definition *A nonnegative function $u\in C_{loc}([0,T);L^1_{loc}(R^N))\cap L^p(0,T;W^{1,p}_{loc}(R^N))$ is said to be a weak solution of* (1.1) *if u satisfies*

$$\int_0^T\int_{R^N}\{u\phi_t-|\nabla u|^{p-2}\nabla u\nabla\phi+u^q\phi\}\mathrm{d}x\mathrm{d}t+\int_{R^N}u_0(x)\phi(x,0)\mathrm{d}x=0, \tag{1.5}$$

for all $\phi\in C_0^\infty(R^N\times[0,T))$.

2. Proof of Theorem 1

To prove Theorem 1, let us introduce the similarity representation of the solution u of (1.1). Set

$$v(\xi,\tau)=(T-t)^{\frac{1}{q-1}}u(x,t), \tag{2.1}$$

with $\xi=\frac{x}{(T-t)^m}\in R^N,\tau=-\ln(\frac{T-t}{T})\in[0,\infty),m=\frac{q-p+1}{p(q-1)}>0$.

Direct calculation leads to the following Cauchy problem for v

$$\begin{cases} v_\tau=\mathrm{div}(|\ \nabla v\ |^{p-2}\ \nabla v)-m\xi\cdot\nabla v-\frac{v}{q-1}+v^q, & (\xi,\tau)\in R^N\times(0,\infty),\\ v(\xi,0)=T^{\frac{1}{q-1}}u_0(\xi T^m), & \xi\in R^N. \end{cases} \tag{2.2}$$

Consequently, v exists globally for all $\tau\in[0,+\infty)$. Hence to prove our theorem, it is sufficient to show that the solution v to (2.2) is bounded for all $\tau\in[0,+\infty)$.

Let us first consider, for fixed numbers $\rho>0,\ Q>0,\ \tau_0\ge0$, the initial-boundary problem

$$\begin{cases} \tilde v_\tau=\mathrm{div}(|\ \nabla\tilde v\ |^{p-2}\ \nabla\tilde v)-m\xi\nabla\tilde v-\frac{\tilde v}{q-1}+\tilde v^q, & (\xi,\tau)\in B(0,2\rho)\times(\tau_0,\infty),\\ \tilde v(\xi,\tau)=0, & (\xi,\tau)\in\partial B(0,2\rho)\times(\tau_0,\infty),\\ \tilde v(\xi,\tau_0)=\chi Q, & \xi\in B(0,2\rho), \end{cases} \tag{2.3}$$