J. Partial Diff. Eqs. **20**(2007), 247–251

Vol.20 No.3

SPACE-TIME ESTIMATE TO HEAT EQUATION

Wang Yidong and Jiang Lingyu
(Department of Mathematics, Central University of Finance and Economics
Beijing 100081, China)
(E-mail: wangyidong@cufe.edu.cn)
(Received Jun. 17, 2006)

Abstract In this article, we prove the Strichartz type estimate for the solutions of linear heat equation with initial data in Hardy space $\mathcal{H}^1(\mathbf{R}^d)$. As an application, we obtain the full space-time estimate to the solutions of heat equation with initial data in $L^p(\mathbf{R}^d)$ for $1 < p < \infty$.
Key Words Strichartz estimate; heat equation; Hardy space.
2000 MR Subject Classification 35, 42.
Chinese Library Classification O175.27.

1. Introduction

In this article, we are interested in the space-time estimate for the solution operator: $u \overset{\text{def}}{=} e^{\nu t\Delta}f$, which can also be written as the solution of the following linear heat equation

$$\partial_t u - \nu\Delta u = 0, \qquad u(t=0,x) = f(x). \tag{1.1}$$

When we restrict (1.1) on $Q_T \overset{\text{def}}{=} [0,T]\times\mathbf{R}^d$, it was proved by Ladyzhenskaya *et al.* in [1] that

$$\|e^{\nu t\Delta}f\|_{L^q([0,T];L^p_x)} \lesssim \nu^{-\frac{1}{q}}\|f\|_{L^r}, \tag{1.2}$$

for q,p satisfying $\frac{1}{q} = (\frac{1}{r}-\frac{1}{p})\frac{d}{2}$ for $p \geq r > 1$. The main idea of the proof is to apply Marcinkiewicz interpolation theorem [2] together with a weak L^1 to $L^m([0,T];L^n)$ estimate for $e^{\nu t\Delta}$, where m,n satisfy $\frac{1}{m} = (1-\frac{1}{n})\frac{d}{2}$, and the trivial estimate that $\|e^{\nu t\Delta}f\|_{L^\infty([0,T]\times\mathbf{R}^d)} \leq \|f\|_{L^\infty}$. Similar estimate was also obtained by Giga in [3], where the author used this type of estimate to study sorts of nonlinear parabolic equations and singular set of weak solutions to 3-D incompressible Navier-Stokes equations.

However to our knowledge, the end case of (1.2) for $f \in \mathcal{H}^1(\mathbf{R}^d)$ and $T = \infty$ seems open. Motivated by [4], we are going to prove a full space-time estimate for heat operator, which in particular covers the above two cases. Now we present the main result of this paper:

Theorem 1.1 *Let $p>1$, q be determined by p via $\frac{1}{q}=(1-\frac{1}{p})\frac{d}{2}$. Let $f\in\mathcal{H}^1(\mathbf{R}^d)$, the standard Hardy space* ([2]). *Then there holds*

$$\|e^{\nu t\Delta}f\|_{L^q_t(L^p_x)}\lesssim \nu^{-\frac{1}{q}}\|f\|_{\mathcal{H}^1}.$$

Consequently, let $1<r,q,p<\infty$ with $\frac{1}{q}=(\frac{1}{r}-\frac{1}{p})\frac{d}{2}$ for $p\geq r>1$; and $f\in L^r(\mathbf{R}^d)$, there holds

$$\|e^{\nu t\Delta}f\|_{L^q_t(L^p_x)}\lesssim \nu^{-\frac{1}{q}}\|f\|_{L^r}.$$

2. The Proof of Theorem 1.1

We start with the proof of Theorem 1.1 by the following Lemma:

Lemma 2.1 *Let $0<r\leq p$, let $f\in\mathcal{H}^r(\mathbf{R}^d)$. Then one has*

$$\begin{aligned}&\|e^{\nu t\Delta}f\|_{L^p_x}\lesssim(\nu t)^{(\frac{1}{p}-\frac{1}{r})\frac{d}{2}}\|f\|_{L^r},\quad \text{if}\quad r>1;\\&\|e^{\nu t\Delta}f\|_{\mathcal{H}^p_x}\lesssim(\nu t)^{(\frac{1}{p}-\frac{1}{r})\frac{d}{2}}\|f\|_{\mathcal{H}^r},\quad \text{if}\quad r\leq 1,\end{aligned}$$

where $\|f\|_{\mathcal{H}^r}$ denotes the Hardy norm of $f(x)$.

Proof of Lemma 2.1 In fact, when $r>1$, we get by applying Young's inequality that

$$\begin{aligned}\|e^{\nu t\Delta}f\|_{L^p_x}&=\frac{1}{(4\pi\nu t)^{\frac{d}{2}}}\|\int_{\mathbf{R}^d}e^{-\frac{|x-y|^2}{4\nu t}}f(y)\,\mathrm{d}y\|_{L^p_x}\\&\lesssim(4\nu t)^{(\frac{1}{p}-\frac{1}{r})\frac{d}{2}}\|f\|_{L^r}.\end{aligned}$$

Whereas when $r\leq 1$, we denote $M^*f(x)\stackrel{\text{def}}{=}\sup_{|x-y|\leq\sqrt{t}}|(f*K_{\sqrt{t}})(y)|$, with $K_{\sqrt{t}}(x)=\frac{1}{(4\pi\nu t)^{\frac{d}{2}}}e^{-\frac{|x|^2}{4\nu t}}$ (see P. 92 of [2]). Then one has

$$\begin{aligned}|e^{\nu t\Delta}f(x)|^r&=|(f*K_{\sqrt{t}})(x)|^r\leq\min_{|x-y|\leq\sqrt{t}}\Big[M^*f(y)\Big]^r\\&\lesssim\frac{1}{t^{\frac{d}{2}}}\int_{|x-y|\leq\sqrt{t}}|M^*f(y)|^r\,\mathrm{d}y\lesssim t^{-\frac{d}{2}}\|M^*f\|^r_{\mathcal{H}^r}\lesssim t^{-\frac{d}{2}}\|f\|^r_{\mathcal{H}^r},\end{aligned}$$

from which, we infer

$$\|e^{\nu t\Delta}f\|_{L^\infty_x}\lesssim t^{-\frac{d}{2r}}\|f\|_{\mathcal{H}^r}.\tag{2.1}$$

On the other hand, we deduce from the definition of $\mathcal{H}^r(\mathbf{R}^d)$ that

$$\|e^{\nu t\Delta}f\|_{\mathcal{H}^r_x}\leq\sup_{t>0}\|e^{\nu t\Delta}f\|_{\mathcal{H}^r_x}\leq\|f\|_{\mathcal{H}^r}.\tag{2.2}$$