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EXISTENCE OF POSITIVE ENTIRE SOLUTIONS FOR POLYHARMONIC EQUATIONS AND SYSTEMS

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Dedicated to Professor Chang Kung-Ching on the occasion of his 70th birthday

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Abstract In this paper, the existence results of positive entire solutions for supercritical polyharmonic equations and system are given.

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1. Introduction

This paper is devoted to the study of the existence of positive entire solutions for polyharmonic equation

$$(-\Delta)^m u = u^p,\ u > 0 \text{ in } \mathbb{R}^N,\ N > 2m. \tag{1.1}$$

and the polyharmonic systems

$$\begin{cases} (-\Delta)^m u = v^q, & u > 0, \\ (-\Delta)^m v = u^p, & v > 0 \end{cases} \text{ in } \mathbb{R}^{\mathbb{N}},\ N > 2m. \tag{1.2}$$

It is known that the Sobolev's exponent $p^* = \frac{N+2m}{N-2m}$ is critical for the existence of positive solutions of equation (1.1). The following results are known (see [1, 2]):

(1) If $0 < p \leq 1$, then (1.1) has no bounded solutions;

(2) If $1 < p < p^*$, then (1.1) has no solutions;

(3) If $p = p^*$ then (1.1) has a family of solutions:

$$u(x) = C_{N,m}\left(\frac{\epsilon}{\epsilon^2 + |x - x_0|^2}\right)^{\frac{N-2m}{2}}, \tag{1.3}$$

where $\epsilon > 0$ and $x_0 \in \mathbb{R}^N$. Moreover, there are no other solutions.

When the problem (1.2) with $m = 1$ is considered, the system (1.2) becomes the well known Lane-Emden system, namely

$$\begin{cases} -\Delta u = v^q, & u > 0, \\ -\Delta v = u^p, & v > 0, \end{cases} \quad \text{in } \mathbb{R}^{\mathbb{N}}. \tag{1.4}$$

In this case, the dividing line between the existence and non-existence of positive solution (u, v) defined in the whole of $\mathbb{R}^N$ is the so called critical hyperbola introduced independently in the work of Clement-deFigueiredo-Mitidier [3] and Peletier-Vorst [4]. This hyperbola is defined by

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-2}{N}, \quad p, q > 0. \tag{1.5}$$

In analogy with the scalar case one may conjecture that (1.4) has no positive solutions defined in the whole of $\mathbb{R}^N$ if p, q satisfy

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \quad p, q > 0. \tag{1.6}$$

Although this conjecture has not been settled so far, it was shown in [5], [6] that if p, q satisfy (1.6), then (1.4) has no nontrivial radial positive solution of class C^2. This result is sharp as far as the critical hyperbola is concerned. Indeed, suppose that $p, q > 0$ and that

$$\frac{1}{p+1} + \frac{1}{q+1} \le \frac{N-2}{N}, p, q > 0. \tag{1.7}$$

Serrin-Zou [7] showed that there exist infinitely many pairs $(\xi, \eta) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that (1.4) admits a positive radial solution (u, v) on $\mathbb{R}^N$ with central values $u(0) = \xi, v(0) = \eta$.

Later, the above existence result was generalized by Serrin-Zou [8] for the general Hamiltonian system of the form:

$$\begin{cases} -\Delta u = & H_v(u, v), \\ -\Delta v = & H_u(u, v), \end{cases} \quad \text{in } \mathbb{R}^{\mathbb{N}}. \tag{1.8}$$

For $m > 1$, the following are known (see [1], [9]):

(4) If $N > 2m$, $p, q \ge 0$ are such that

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2m}{N},$$