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GLOBAL SOLVABILITY FOR A NONLINEAR SEMI-STATIC MAXWELL'S EQUATION

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Dedicated to Professor Jiang Lishang on the occasion of his 70th birthday

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Abstract In this paper we study a nonlinear Maxwell's system in a highly conductive medium in which the displacement current is neglected. The magnetic field **H** satisfies a quasilinear evolution system:

$$\mathbf{H}_t + \nabla \times [r(x, t, |\mathbf{H}|, |\nabla \times \mathbf{H}|)\nabla \times \mathbf{H}] = \mathbf{F}(x, t, \mathbf{H}),$$

where the resistivity r is assumed to depend upon the strengths of electric and magnetic fields while the internal magnetic current **F** depends upon the magnetic field. It is shown that under appropriate structure conditions for r and **F** the above nonlinear system subject to appropriate initial-boundary conditions has a unique global solution.

Key Words Nonlinear Maxwell's Equations; Bean-like critical-state model; Global existence and uniqueness.

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1. Introduction

Recently, Bean's critical-state model in the superconductivity theory ([1]) has received considerable attention (see [2-9]). The importance of this model is that it gives a criterion in experiments for searching various superconductor materials ([5, 10]). A simple approximation of the model is given by the following system for the magnetic field **H** with large p:

$$\mu\mathbf{H}_t + \nabla \times \left[|\nabla \times \mathbf{H}|^{p-2}\nabla \times \mathbf{H}\right] = \mathbf{F}(x, t), \qquad p \geq 2,$$

where μ represents the magnetic permeability while $r = |\nabla \times \mathbf{H}|^{p-2}$ is the resistivity (see [6, 8] for the justification). Hereafter, a bold letter represents a vector or vector function in R^3.

However, Bean's model is not quite compatible with some experiments. A Bean-like model is proposed in [2] with a more general resistivity r (also see [4] for the model). This motivates us to study the following general system:

$$\mu \mathbf{H}_t + \nabla \times [r(x,t,|\mathbf{H}|,|\nabla \times \mathbf{H}|)\nabla \times \mathbf{H}] = \mathbf{F}(x,t,\mathbf{H}), \quad \text{in } Q_T, \tag{1.1}$$

where $Q_T = \Omega \times (0,T]$ and Ω is a bounded simply-connected domain in R^3 with C^1-boundary. The system (1.1) is also encountered in other industrial applications when materials are conductive, anisotropic and nonlinear ([11]). For this type of materials the electric conductivity σ may depend on the strengths of electric and magnetic fields, i.e. $\sigma = \sigma(x,t,|\mathbf{E}|,|\mathbf{H}|)$. By neglecting the displacement current, we see from Ampere's law that

$$\mathbf{E}(x,t) = r\nabla \times \mathbf{H}(x,t),$$

where $r = \frac{1}{\sigma}$ represents the resistivity of the material, which depends on the electric field $\mathbf{E}$ and $\mathbf{H}$. Suppose there exists an internal magnetic current which depends on the magnetic field, denoted by $\mathbf{F}(x,t,\mathbf{H})$. Then, classical Maxwell's equations for $\mathbf{E}$ and $\mathbf{H}$ (see [12]) can be expressed by a single evolution system (1.1). The purpose of this paper is to study the solvability for Eq. (1.1) subject to appropriate initial and boundary conditions.

The major challenge for the solvability to the system (1.1) arises from the fact that $\nabla \cdot \mathbf{H}$ is unknown and the operator $L[\mathbf{H}] := \nabla \times [r\nabla \times \mathbf{H}]$ does not satisfy the ellipticity condition even for a bounded r with a positive lower bound. Considerable progress has been made for the global solvability to this type of equations. For the linear case where $r = r(x,t)$, the author of [13] derived optimal regularity of $\mathbf{H}(x,t)$ when $r(x,t)$ is continuous or Hölder continuous. Moreover, for the steady-state case, it is shown that the weak solution was Hölder continuous if $r = r(x)$ is only assumed to be bounded with positive lower bound (see [14]). In [8, 9], the authors studied the following p-curl type of system:

$$\mathbf{H}_t + \nabla \times [|\nabla \times \mathbf{H}|^{p-2}\nabla \times \mathbf{H}] = \mathbf{F}(x,t)$$

subject to appropriate initial and boundary conditions. The global existence and uniqueness are established for any $p > 1$. Moreover, it is shown that the limit solution as $p \to \infty$ solves Bean's critical-state model in three-space dimensions (see [8, 9] for details). More recently, the author of [15] studied the following semilinear system:

$$\mathbf{H}_t + \nabla \times [r(x,t)\nabla \times \mathbf{H}] = f(|\mathbf{H}|)\mathbf{H}$$

subject to appropriate initial-boundary conditions. Under the certain growth condition for $f(s)$, the author proved that the problem has a global solution. Moreover, it is shown that the solution blows up in finite time if $f(s)$ grows suitably fast with respect to s. Typical examples in [15] include $\mathbf{F}(\mathbf{H}) = (1 - |\mathbf{H}|^p)\mathbf{H}$ and $\mathbf{F}(\mathbf{H}) = e^{-|\mathbf{H}|}\mathbf{H}$ for the global existence while $\mathbf{F}(\mathbf{H}) = |\mathbf{H}|^p\mathbf{H}$ and $\mathbf{F}(\mathbf{H}) = e^{|\mathbf{H}|}\mathbf{H}$ for the blowup case.