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COMPLETE REDUCIBILITY FOR QUASILINEAR HYPERBOLIC SYSTEMS*

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Dedicated to Professor Jiang Lishang on the occasion of his 70th birthday

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Abstract In this paper we present a necessary and sufficient condition to guarantee the complete reducibility for quasilinear hyperbolic systems and give some examples.

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1. Introduction

In [1] we have introduced the concept of the completely reducible quasilinear hyperbolic system and discussed the singularity caused by eigenvectors for this kind of system in the case of constant eigenvalues. In this paper we will present a method for checking if a given quasilinear strictly hyperbolic system is completely reducible or not, and give some examples.

2. A Necessary and Sufficient Condition for a Quasilinear Strictly Hyperbolic System Being Completely Reducible

Consider the following first order quasilinear strictly hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{2.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x) and $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth entries $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

By strict hyperbolicity, on the domain under consideration $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u),\ \lambda_2(u),\ \cdots,\ \lambda_n(u). \tag{2.2}$$

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For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{2.3}$$

All $\lambda_i(u)$, $l_i(u)$ and $r_i(u)$ $(i = 1, \cdots, n)$ have the same regularity as $A(u)$. Without loss of generality, we assume that

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1, \cdots, n), \tag{2.4}$$

where δ_{ij} denotes the Kronecker's symbol.

System (1.1) can be equivalently reduced to the following characteristic form

$$l_i(u)\Big(\frac{\partial u}{\partial t} + \lambda_i(u)\frac{\partial u}{\partial x}\Big) = 0 \quad (i = 1, \cdots, n). \tag{2.5}$$

For $i = 1, \cdots, n$, the i-th equation in (2.5) contains only the directional derivative of u with respect to t along the i-th characteristic direction $\frac{dx}{dt} = \lambda_i(u)$.

By the definition given in [1], system (2.1) is m-step (globally) completely reducible, if there is a global diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$

$$u = u(\tilde{u}) \tag{2.6}$$

such that the corresponding system for $\tilde{u}$

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{A}(\tilde{u})\frac{\partial \tilde{u}}{\partial x} = 0 \tag{2.7}$$

has the following standard form:

$$\tilde{A}(\tilde{u}) = \begin{pmatrix} \tilde{\Lambda}^{(1)}(\tilde{u}) & & & \\ \tilde{A}_{21}(\tilde{u}) & \tilde{\Lambda}^{(2)}(\tilde{u}) & & \\ \vdots & & \ddots & \\ \tilde{A}_{m1}(\tilde{u}) & \cdots \tilde{A}_{m,m-1}(\tilde{u}) & & \tilde{\Lambda}^{(m)}(\tilde{u}) \end{pmatrix}, \tag{2.8}$$

where $\tilde{\Lambda}^{(a)}(\tilde{u})$ $(a = 1, \cdots, m)$ are diagonal matrices, the entries of which are given by $\tilde{\lambda}_i(\tilde{u}) = \lambda_i(u(\tilde{u}))$ $(i = 1, \cdots, n)$ respectively. If this diffeomorphism (2.6) is only valid in a local domain, system (2.1) is called to be m-step locally completely reducible. If there is no such diffeomorphism (2.6) even in the local sense, system (2.1) is non-completely reducible.

Without loss of generality, in what follows we consider only the 2-step completely reducible case.

By definition, under diffeomorphism (2.6), a 2-step completely reducible quasilinear strictly hyperbolic system (2.1) can be reduced to the following standard form

$$\begin{cases} \dfrac{\partial \tilde{u}^{(1)}}{\partial t} + \tilde{\Lambda}^{(1)}(\tilde{u})\dfrac{\partial \tilde{u}^{(1)}}{\partial x} = 0, \\ \dfrac{\partial \tilde{u}^{(2)}}{\partial t} + \tilde{\Lambda}^{(2)}(\tilde{u})\dfrac{\partial \tilde{u}^{(2)}}{\partial x} + \tilde{A}_{21}(\tilde{u})\dfrac{\partial \tilde{u}^{(1)}}{\partial x} = 0, \end{cases} \tag{2.9}$$