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BLOW-UP OF SOLUTIONS TO QUASILINEAR PARABOLIC EQUATIONS*

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Abstract This paper investigates the qualitative properties of solutions to certain quasilinear parabolic equations. Under appropriate conditions, we obtain that the solution either exists globally or blows up in finite time by making use of the energy method and subsolution techniques. We find out that the behavior of solution heavily depends on the sign and the growth rate of the nonlinear reaction term and the nonlinear flux through boundary at infinity.
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1. Introduction

In this paper we consider the following quasilinear parabolic equations with nonlinear terms

$$\begin{aligned} u_t - \Delta_p u + f(u) = 0 \quad & \text{in} \quad \Omega \times (0,T), \\ \nabla_p u \cdot \nu + g(u) = 0 \quad & \text{on} \quad \Gamma \times [0,T], \\ u(x,0) = u_0(x) \quad & \text{in} \quad \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N, N \geq 1$, with sufficiently smooth boundary Γ, ν denotes the unit outward normal vector. Δ_p denotes the p-Laplacian with $p > 2$, ∇_p is the differential operator associated with Δ_p, i.e., $\Delta_p u \equiv \text{div}(\nabla_p u)$, $\nabla_p u \equiv |\nabla u|^{p-2}\nabla u$. Functions $f, g : \mathbb{R} \to \mathbb{R}$ are smooth functions satisfying growth conditions, to be stated in Section 2.

The equation in (1.1) is a class of quasilinear parabolic equations and appears in the relevant theory of non-Newtonian fluids [1].

In last three decades, the problem like (1.1) has been well investigated by many authors, for example, see [1-16] and the references cited therein. However, most of these

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literatures works deal with the local existence, regularity and the decay of solutions [1-7, 11-13, 15] and only a few of them concern the blow-up of solutions [8-10, 14, 16]. In [10, 14, 16], p-Laplacian equations with Dirichlet boundary condition has been considered and some sufficient conditions are obtained to ensure that the solutions blow up in finite time.

Galaktionov [8, 9] has studied the following Cauchy problem

$$\begin{aligned} u_t - \nabla\cdot(|\nabla u|^\sigma \nabla u) &= u^p \quad \text{for} \quad (x,t)\in\mathbb{R}^N\times(0,+\infty),\\ u(x,0) &= u_0(x) \qquad \text{for} \quad x\in\mathbb{R}^N \end{aligned} \tag{1.2}$$

with $\sigma > 0$. He obtains that: (i) In case of $1 < p \le 1+\sigma+(\sigma+2)/N$, then all solutions of the problem (1.2) blow up in finite time; (ii) In case of $p > 1+\sigma+(\sigma+2)/N$, then u blows up in finite time for large initial data $u_0(x)$ or exists globally for small data $u_0(x)$.

Recently, Chipot and Filo [17] consider the following model problem

$$\begin{aligned} &u_t = (a(u_x))_x, && 0<x<1,\ t>0,\\ &u_x|_{x=0} = 0,\ a(u_x)|_{x=0} = u^\alpha|_{x=0}, && t>0,\\ &u(x,0) = u_0(x), && 0\le x\le 1, \end{aligned} \tag{1.3}$$

where $a(\xi)\in C^4(\mathbb{R})$ satisfies $a(\xi) = |\xi|^{p-2}\xi$ if $|\xi|\ge\eta>0$, $a'(\xi)>0$ and $|a(\xi)|\le|\xi|^{p-1}$ for all $\xi\in\mathbb{R}$; $0<\eta\ll 1$ is any given positive constant. In case of $p>2$, they show that the solution of (1.3) either exist globally if $\alpha\le 1$ or blows up in finite time if $\alpha>1$ for large initial data.

The purpose of this paper is to investigate how the nonlinear reaction term and the nonlinear flux through boundary affect the solution of the problem (1.1). We obtain some balance conditions which cause the solution either exists globally or blows up in finite time. We point out that the behavior of solution is dominated by the sign and the growth rate of the nonlinear terms f and g at infinity.

To illustrate our results, we consider the power-type nonlinear case:

$$\lim_{s\to\infty} f(s)/s^\alpha = C_\alpha, \quad \lim_{s\to\infty} g(s)/s^\beta = C_\beta$$

for $0<\alpha<((p-1)(N+2)+2)/N$, $0<\beta<(p-1)(N+2)/N$ and some constants $C_\alpha, C_\beta\in\mathbb{R}\backslash\{0\}$. In this case, our results can be described as follows:

1. $C_\alpha>0, C_\beta>0$, u exists globally.
2. $C_\alpha<0, C_\beta<0$,
   (a) $\max\{\alpha,\beta\}\le 1$, u exists globally;
   (b) $\alpha>1$ or $\beta>1$, u blows up in finite time.