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LIOUVILLE-TYPE THEOREMS FOR SEMILINEAR ELLIPTIC SYSTEMS*

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Abstract Liouville-type theorems for a class of semilinear elliptic systems are considered via the method of moving spheres.

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1. Introduction

In this paper we consider the Lane-Emden system

$$\begin{cases} -\Delta u = v^{\alpha} \\ -\Delta v = u^{\beta} \end{cases} \quad \text{in} \quad \mathbf{R}^N (N \geq 3). \tag{1.1}$$

The question is to determine for which values of the exponents α and β the only non-negative solution (u, v) of (1.1) is $(u, v) = (0, 0)$. The solution here is taken in the classical sense, i.e., $u, v \in C^2(\mathbf{R}^N)$. In the case of a single equation, or its equivalent, the Emden-Fowler equation

$$\Delta u + u^p = 0, \quad u \geq 0 \quad \text{in} \quad \mathbf{R}^N. \tag{1.2}$$

When $1 \leq p < (N+2)/(N-2) (N \geq 3)$, it has been proved in [1] that the only solutions of (1.2) is $u = 0$. In dimension $N = 2$, a similar conclusion holds for $0 \leq p < \infty$. It is also well known that in the critical case, $p = (N+2)/(N-2)$, the problem (1.2) has a two-parameter family of solutions given by

$$u(x) = \Big(\frac{a}{d + |x - \overline{x}|^2}\Big)^{\frac{N-2}{2}}, \tag{1.3}$$

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where $a = (N(N-2))^{\frac{1}{2}}\mu,\ \ d = \mu^2$ with $\mu > 0$ and $\overline{x} \in \mathbf{R}^N$.

The results proved here are the following:

Theorem 1.1 *If $0 < \alpha, \beta \leq (N+2)/(N-2)$, but not both are equal to $(N+2)/(N-2)$, then the only non-negative C^2 solution of the problem (1.1) in $\mathbf{R}^N$ is the trivial one, i.e., $(u, v) = (0, 0)$.*

Theorem 1.2 *If $\alpha = \beta = (N+2)/(N-2)$, then the positive C^2 solution of the problem (1.1) is of the form (1.3).*

There are some related works about the problem (1.1). Figueiredo and Felmer (see [2]) proved Theorem 1.1 by using the moving plane method and a special form of the maximum principle for elliptic systems. Busca and Manásevich obtained a new result (see [3, Theorem 2.1]) by using the same method as in [2]. It allows α and β to reach regions where one of the two exponents is supercritical. In this paper, we shall first introduce the Kelvin transforms and give a different proof of Theorem 1.1 by using the method of moving spheres. This approach was suggested in [4], while Li and Zhang had made significant simplifications prove some Liouville theorems for a single equation in [5]. We extend the approach to the elliptic systems and don't need the maximum principle for elliptic systems. Moreover, the exact form of positive solution is proved when the two exponents are both critical, i.e., Theorem 1.2. If we can find a proper transforms instead of the Kelvin transforms, we suspect that Theorem 2.1 in [3] can also be proved via the method of moving spheres. We leave this to the interested reader. Furthermore, we can easily obtain a corollary of Theorem 1.1 and 1.2.

Corollary 1.3 *For the following elliptic system consisting of m equations*

$$\begin{cases} -\Delta u_1 = u_2^{\alpha_1} \\ -\Delta u_2 = u_3^{\alpha_2} \\ \quad\quad \vdots \\ -\Delta u_m = u_1^{\alpha_m} \end{cases} \quad in \quad \mathbf{R}^N (N \geq 3) \tag{1.4}$$

where $0 < \alpha_1, \ldots, \alpha_m \leq (N+2)/(N-2),\ \ m \geq 2$, Theorem 1.1 and Theorem 1.2 still hold.

Let us emphasize that considerable attention has been drawn to Liouville-type results and existence of positive solutions for general nonlinear elliptic equations and systems, and that numerous related works are devoted to some of its variants, such as more general quasilinear operators, and domains. We refer the interested reader to [6-10], and some of the references therein.

2. Preliminaries and Moving Spheres

To prove the Liouville-type theorems, we shall use the method of moving spheres. We first prove a number of lemmas as follows. For $x \in \mathbf{R}^N$ and $\lambda > 0$, let us introduce the Kelvin transforms

$$u_{x,\lambda}(y) = \frac{\lambda^{N-2}}{|y-x|^{N-2}} u\Big(x + \frac{\lambda^2 (y-x)}{|y-x|^2}\Big), \quad v_{x,\lambda}(y) = \frac{\lambda^{N-2}}{|y-x|^{N-2}} v\Big(x + \frac{\lambda^2 (y-x)}{|y-x|^2}\Big)$$