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# THE CAUCHY PROBLEM OF NONLINEAR SCHRÖDINGER-BOUSSINESQ EQUATIONS IN $H^s(R^d)$

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**Abstract** In this paper, the local well posedness and global well posedness of solutions for the initial value problem (IVP) of nonlinear Schrödinger-Boussinesq equations is considered in $H^s(R^d)$ by resorting Besov spaces, where real number $s \geq 0$.
**Key Words** Schrödinger-Boussinesq equation; global solutions in Besov spaces.
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## 1. Introduction

We consider the existence and uniqueness of the local solutions and global solutions for the following initial value problem (IVP) of nonlinear Schrödinger-Boussinesq equations

$$i\epsilon_t + \Delta\epsilon - n\epsilon - A|\epsilon|^p\epsilon = 0, \tag{1.1}$$

$$n_{tt} - \Delta(n - \Delta n + Bn^{K+1} + |\epsilon|^2) = 0, \quad x \in R^d, t \in R, \tag{1.2}$$

$$\epsilon(x,0) = \epsilon_0(x), \quad n(x,0) = n_0(x), \quad n_t(x,0) = \Delta\phi_0(x), \quad x \in R^d, \tag{1.3}$$

where $A$ and $B$ are constants, $K$ is a positive integer, real number $p > 0$; $\epsilon$ and $\epsilon_0$ are complex functions; $n$, $n_0$ and $\phi_0$ are real functions; $\Delta$ is Laplacian operator in $R^d$.

The nonlinear Schrödinger (NLS) equation models a wide range of physical phenomena including self-focusing of optical beams in nonlinear media, propagation of Langmuir waves in plasmas, etc. (see [1] and the references therein). Boussinesq equation as a model of long waves is derived in the studies of the propagation of long waves on the surface of shallow water[2], the nonlinear string [3] and the shape-memory alloys[4], etc. The nonlinear Schrödinger-Boussinesq equations (1.1)(1.2) is considered as a model of interactions between short and intermediate long waves, which is derived in describing the dynamics of Langmuir soliton formation and interaction in a plasma [5-7] and diatomic lattice system [8], etc.

The Solitary wave solutions and integrability of nonlinear Schrödinger-Boussinesq equations has been considered by several authors, see [5, 6, 9] and the references therein. In [10] Guo established the existence and uniqueness of global solution for IVP (1.1)–(1.3) in $H^k$ (integer $k \geq 4$) with $d = 1$ and $A = 0$. In [11] the existence and uniqueness of global solution for Cauchy problem of dissipative Schrödinger-Boussinesq equations in $H^k$ (integer $k \geq 4$) with $d = 3$ is proved by Guo and Shen. For damped and dissipative Schrödinger-Boussinesq equations with initial boundary value, the existence of global attractors and the finiteness of the Hausdorff and the fractal dimensions of the attractor is established by Guo and Chen ([12], d=1) and Li and Chen ([13], $d \leq 3$), respectively.

In this paper, the local well-posedness in $H^s$, the conservation of energy and the global well-posedness in $H^s$ (real number $s \geq 1$ and $d = 1, 2, 3$) of IVP (1.1)–(1.3) is proved.

**Definition 1**(admissible pair) *The pair $(q, r)$ is admissible if $\frac{2}{q} = d(\frac{1}{2} - \frac{1}{r})$; $2 \leq r \leq \infty$ for $d = 1$, $2 \leq r \leq \infty$ for $d = 2$, $2 \leq r < \frac{2d}{d-2}$ for $d \geq 3$.*

**Definition 2***(condition $P(m)$) For a positive integer $m$, it is called that $p$ satisfies the condition $P(m)$ if either $p$ is an even integer, or $p$ is not an even integer and $p + 1 > m$.*

The main theorems of this paper are stated as follows.

**Theorem 1** *Suppose that $\epsilon_0, n_0, \phi_0 \in H^s(R^d)$, $0 \leq s < \frac{d}{2}$, $K$ is an integer, $p$ satisfies the condition $P([s] + 1)$, $0 < p$, $K \leq \frac{4}{d-2s}$; then for any admissible pair $(q, r)$, there exists $T = T(\epsilon_0, n_0, \phi_0) > 0$ and a unique solution $(\epsilon, n)$ of IVP (1.1)–(1.3) such that*

$$\epsilon, n, (-\Delta)^{-1} n_t \in L^q\left(0, T; B^s_{r,2}(R^d)\right) \cap C\left([0, T]; H^s(R^d)\right)$$

*Moreover, this solution has the following additional properties.*

(I) *Let $p$, $K < \frac{4}{d-2s}$. If $\epsilon_{0j}, n_{0j}, \phi_{0j}$ are sequences in $H^s(R^d)$ with $(\epsilon_{0j}, n_{0j}, \phi_{0j}) \to (\epsilon_0, n_0, \phi_0)$, then there exists $\tilde{T} = \tilde{T}(\epsilon_0, n_0, \phi_0) \in (0, T]$, such that the solutions $(\epsilon_j, n_j) \to (\epsilon, n)$ and $(-\Delta)^{-1}\partial_t n_j \to (-\Delta)^{-1} n_t$ in $L^q\left(0, \tilde{T}; L^r(R^d)\right)$, where $(\epsilon_j, n_j)$ are solutions of IVP (1.1)–(1.3) with $(\epsilon_0, n_0, \phi_0)$ replaced by $(\epsilon_{0j}, n_{0j}, \phi_{0j})$. If $s \geq 1$, then $(\epsilon_j, n_j) \to (\epsilon, n)$ and $(-\Delta)^{-1}\partial_t n_j \to (-\Delta)^{-1} n_t$ in $C\left([0, \tilde{T}]; H^{s-1}(R^d)\right) \cap L^q\left(0, \tilde{T}; B^{s-1}_{r,2}\right)$. Moreover, if $p$ satisfies the condition $P([s] + 2)$, then $(\epsilon_j, n_j) \to (\epsilon, n)$ and $(-\Delta)^{-1}\partial_t n_j \to (-\Delta)^{-1} n_t$ in $C\left([0, \tilde{T}]; H^s(R^d)\right) \cap L^q\left(0, \tilde{T}; B^s_{r,2}\right)$.*

(II) *There exists $T^\star = T^\star(\epsilon_0, n_0, \phi_0) > 0$ such that the solution $\epsilon, n, (-\Delta)^{-1} n_t \in C\left([0, T^\star); H^s(R^d)\right) \cap L^q_{loc}\left(0, T^\star; B^s_{r,2}(R^d)\right)$. If $T^\star < \infty$, then*

$$\lim_{t \to T^\star} \left\{ \|(-\Delta)^{\frac{s}{2}} \epsilon(\cdot, t)\|_{L^2} + \|(-\Delta)^{\frac{s}{2}} n(\cdot, t)\|_{L^2} + \|(-\Delta)^{\frac{s-2}{2}} n_t(\cdot, t)\|_{L^2} \right\} = +\infty.$$