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ASYMPTOTIC BEHAVIOR OF THE NONLINEAR PARABOLIC EQUATIONS

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Abstract This paper is concerned with the large time behavior for solutions of the nonlinear parabolic equations in whole spaces $\mathbf{R}^n$. The spectral decomposition methods of Laplace operator are applied and it is proved that if the initial data $u_0 \in L^2 \cap L^r$ for $1 \le r \le 2$, then the solutions decay in L^2 norm at $t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}$. The decay rates are optimal in the sense that they coincide with the decay rates of the solutions to the heat equations with the same initial data.

Key Words L^2 decay; spectral decomposition; nonlinear parabolic equation.
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1. Introduction

In this paper we investigate the optimal time decay rate of global solutions to the Cauchy problem of the following nonlinear parabolic equations

$$u_t - \triangle u - \nabla \cdot (|u|\nabla u) = 0, \quad \text{in } \mathbf{R}^n \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0, \quad \text{in } \mathbf{R}^n, \quad n \ge 3, \tag{1.2}$$

which appear to be relevant in the theory of the viscous incompressible non-Newtonian fluids (refer to [1-3]).

As for the generalized nonlinear parabolic equations

$$u_t - \triangle u + F(u, D_x u, D_x^2 u) = 0, \quad x \in \mathbf{R}^n,\ t > 0, \tag{1.3}$$

there is an extensive literature on the well-posedness and large time behavior of solutions. The Cauchy problem of (1.3) has been studied by many authors and a lot of good results have been obtained (a complete literature in this direction is beyond the scop of this paper, however, we want to mention [1-16] and the references sited therein). To go directly to the main points of the present paper, in what follows we only review some former results which are closely related to our main results. Escobedo and Zuazua [4, 5] discussed the large time behavior of the global smooth solutions of the diffusion-convection equations in which the nonlinear term f has the type of $(a\cdot\nabla)u$ (a is a constant vector). They mainly applied the Fourier splitting methods which were first developed in scalar parabolic conversation laws and Navier-Stokes equations by Schonbek [10, 11]. Ponce [9], Zheng and Chen [16] studied the global existence of the solutions with the small initial data. Zhang [13] investigated both the L^2 and L^∞ decay of the solutions of the nonlinear parabolic system of filtration type in one-dimension where the nonlinear term f has the types of $(grad\ \varphi(u))_{xx}$ and $(R(u))_x+(\varphi(u))_x$ (here the $\varphi(u)$ is the scalar function of the real variable $u\in\mathbf{R}^1$ and R is an operator with the even symbol). Moreover, Zhang [14] recently discussed the $L^p\,(p=1,2\ or\ \infty)$ decay of the solutions of nonlinear dissipative equations with the dissipative term $N(\varphi,\nabla\varphi)$ which satisfies $\int_{\mathbf{R}^n}\varphi^{2m-1}N(\varphi,\nabla\varphi)\ dx=0$. Zhao [15] also investigated the large time behavior of the solutions with the nonlinear term f satisfying some growth conditions.

In the present paper, we investigate the time decay problem of the solutions to the Cauchy problem of the nonlinear parabolic equations with nonlinear term $\nabla\cdot(|u|\nabla u)$ which appears to be relevant in the theory of the viscous incompressible non-Newtonian fluids and the existence was first discussed by J. L. Lions [2]. The main difficulty of time decay problem is how to deal with the nonlinear part $\nabla\cdot(|u|\nabla u)$, we will apply the spectral decomposition methods which were first developed in Navier-Stokes equations by Kajikiya and Miyakawa [7], *i.e.* our main tools are the spectral decomposition and fractional powers of the Laplace operator in general L^r $(1<r<\infty)$ spaces. It is well known that the existence of fractional powers of the Laplace operator is guaranteed by the fact that the Laplace operator on $\mathbf{R}^n$ generates a bounded analytic semigroup in each $L^r(\mathbf{R}^n)$ spaces [6-8]. By applying the basic L^p-L^q estimates for the solution of the heat equations, we will show that if the initial data $u_0\in L^2(\mathbf{R}^n)\cap L^r(\mathbf{R}^n)$ for $1\le r\le 2$, then the weak solution decays in L^2 norm at $t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}$, the decay rates are optimal in the sense that they coincide with the decay rates of the solutions to the heat equations with the same initial data.

2. Main Results

Throughout this paper we denote by $L^q(\mathbf{R}^n)$ the usual Lebesgue space with the norm $\|\cdot\|_q$. In particular $\|\cdot\|=\|\cdot\|_2$. $W^{m,p}(\mathbf{R}^n)$ is the usual Sobolev space with the norm $\|\cdot\|_{m,p}$. For a Banach space X, $L^q(0,T;X)$ is the space of all measurable functions $u:(0,T)\mapsto X$ with the norm $\|u\|^q_{L^q(0,T;X)}=\int_0^T\|u\|^q_X dt$. When $q=\infty$,