J. Partial Diff. Eqs. **16**(2003), 381–383
©International Academic Publishers Vol.16 No.4

SHORT COMMUNICATION SECTION

AN INITIAL VALUE PROBLEM FOR PARABOLIC MONGE-AMPÈRE EQUATION FROM INVESTMENT THEORY

Wang Guanglie and Lian Songzhe
(Department of Mathematics, Jilin University, Changchun 130023, China)
(E-mail: glwang@public.cc.jl.cn)
(Received Jun. 23, 2003)

Key Words Existence of solution; Optimal portfolio.
2000 MR Subject Classification 35K15, 35K55.
Chinese Library Classification O175.26, O175.29.

The author of [1] raised an optimal investment problem in time interval $[0,T]$, in which the financial market is characterized by the parameters r, b, σ, the attitude of the investor to the risk versus the gain at the final time is described by a utility function $g(y)$, the purpose is to find out an optimal portfolio to maximize the profit of the investor. To this end, in [1] the following initial value problem is derived:

$$\begin{cases} V_s V_{yy} + ryV_yV_{yy} - \theta V_y^2 = 0, \quad V_{yy} < 0, \quad (s,y) \in [0,T) \times \mathbb{R}, \\ V(T,y) = g(y), \quad g'(y) \geq 0, \quad y \in \mathbb{R}, \end{cases} \tag{1}$$

where $V = V(s,y)$ is the unknown function, constants $r \geq 0, \sigma > 0, b-r > 0, \theta = \dfrac{b-r}{\sigma}$, and

$$g(y) = 1 - e^{-\lambda y} \tag{2}$$

is a typical case, where λ is a positive constant. The relation between the optimal investment problem and (1) lies in

Lemma 1 *Suppose (1) admits a classical solution $V(s,y)$ such that the function*

$$\tilde{\pi}(s,y) \stackrel{\text{def}}{=} -\frac{\theta V_y(s,y)}{\sigma V_{yy}(s,y)}, \quad (s,y) \in [0,T] \times \mathbb{R} \tag{3}$$

is Lipschitz continuous in y. Then $V(s,y)$ is the value function of the optimal problem with the optimal portfolio given by

$$\overline{\pi}(t) \equiv \tilde{\pi}(t, \overline{Y}(t)), \quad t \in [s,T], \tag{4}$$

where $\overline{Y}(\cdot)$ *is the solution of*

$$\begin{cases} d\overline{Y}(t) = r\overline{Y}(t) + (b-r)\tilde{\pi}(t, \overline{Y}(t))]dt, \\ +\sigma\tilde{\pi}(t, \overline{Y}(t))dW(t), \quad t \in [s, T], \\ \overline{Y}(s) = y. \end{cases} \tag{5}$$

Which is proved in [1].

The equation in (1) is called parabolic Monge-Ampère equation in [1], which is indeed a nonlinear and un-uniformly parabolic equation. But there is not any existence result for it in [1].

We obtain a general approach to both the solution to (1) and the optimal portfolio to the optimal investment problem, which goes like this:

Let $f(s, y)$ be a smooth function, which is Lipschitz continuous in y. Insert

$$\frac{V_y}{V_{yy}} = f(s, y) \tag{6}$$

into (1), then (1) becomes a Cauchy problem for a homogeneous linear partial differential equation of first order, which is our key observation. And, by the known result, the unique solution of this Cauchy problem is $g(Y(T; s, y))$, where $y = Y(s; s_0, y_0)$ is the solution of the initial value problem for its characteristic equation

$$\begin{cases} \dfrac{dy}{ds} = f(s, y) - ry, \\ y|_{s=s_0} = y_0. \end{cases} \tag{7}$$

Now it is obvious that, in order that the function $g(Y(T; s, y))$ can be the solution to (1), we need and only need that the function $V(s, y) = g(Y(T; s, y))$ satisfies (6), which can be expressed in a formula; since, by the theorem of differentiability of the solution of (7) w.r.t. the initial value, we can calculate $\dfrac{\partial g(Y(T; s, y))}{\partial y}$ and $\dfrac{\partial^2 g(Y(T; s, y))}{\partial y^2}$.

We may summarize the above general approach into the following

Theorem 1 *Suppose* $f(s, y)$ *is a smooth function, which is Lipschitz continuous in* y*. Then (1) will have a solution* $V(s, y)$ *with the property*

$$\tilde{\pi}(s, y) \stackrel{\text{def}}{=} -\frac{\theta V_y(s, y)}{\sigma V_{yy}(s, y)} = -\frac{\theta}{\sigma} f(s, y), \quad (s, y) \in [0, T] \times \mathbb{R} \tag{3}_f$$

if and only if the following condition holds:

$$\frac{g'(Y(T; s, y)) \exp\left\{\displaystyle\int_T^s \left[r - \theta\frac{\partial f}{\partial Y}(\xi, Y(\xi, y))\right] d\xi\right\}}{g''(Y(T; s, y)) + g'(Y(T; s, y))\displaystyle\int_T^s \theta\frac{\partial^2 f}{\partial Y^2}(\tau, Y(\tau; s, y)) \exp\int_T^\tau \left[(r - \theta\frac{\partial f}{\partial Y}(\xi, Y(\xi; s, y))\right] d\xi d\tau} = f(s, y), \tag{8}$$

where $Y(s; s_0, y_0)$ *is defined by the general solution to (6). When (8) is valid, (1) has a solution satisfying* $(3)_f$*, which is*

$$V(s, y) = g(Y(T; s, y)) \tag{9}$$