MORREY REGULARITY OF SOLUTIONS TO DEGENERATE ELLIPTIC EQUATIONS IN $\mathbb{R}^n$

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Abstract In this paper, we study the Morrey regularity of solutions to the degenerate elliptic equation $-(a_{ij}u_x)_x_j = -(f_j)_x_j$ in $\mathbb{R}^n$. For this purpose, we introduce four weighted Morrey spaces in $\mathbb{R}^n$.

Key Words Regularity; degenerate elliptic equations; weighted Morrey spaces.

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1. Introduction

The aim of this paper is to consider in $\mathbb{R}^n$ the equation

$$Lu \equiv -(a_{ij}u_x)_x_j = -(f_j)_x_j,$$

(1)

for which we assume that $a_{ij}(x)$ are symmetry, measurable and there exists $\nu > 0$, such that for all $\xi \in \mathbb{R}^n$ and a.a. $x \in \mathbb{R}^n$,

$$\nu^{-1}\omega(x)|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \nu\omega(x)|\xi|^2,$$

(2)

where $\omega(x)$ belongs to the Muckenhoupt class $A_2$. We also assume that $f_j/\omega \in L^2(\mathbb{R}^n, \omega)$.

Since the middle of the 20th century, people have gotten many results about the equation (1) in the bounded open subset of $\mathbb{R}^n$. And we can also consider the equation in $\mathbb{R}^n$. In [1], S. Leonardi studied in $\mathbb{R}^n$ the equation (1) in the uniformly elliptic case. We will extend the results in [1] to the degenerate case. For this purpose, we will introduce four weighted Morrey spaces in the next section.

2. Preliminaries

We give some definitions first.
Definition 2.1 Let $\omega(x) > 0$, $\omega(x) \in L^1_{\text{loc}}(\mathbb{R}^n)$, $1 < p < +\infty$. We say $\omega(x)$ is an $A_p$ weight, which is denoted by $\omega(x) \in A_p$ if
\[
\sup_Q \left( \frac{1}{|Q|} \int_Q |y|dy \right) \left( \frac{1}{|Q|} \int_Q |y|^{-\frac{1}{p-1}}dy \right)^{p-1} \leq C < +\infty,
\]
where $Q$ is a cube in $\mathbb{R}^n$.

Let $\omega$ be an open set of $\mathbb{R}^n$, $\omega$ be an $A_2$ weight, $1 \leq p < +\infty$. We give the definitions of weighted Lebesgue spaces and weighted Sobolev spaces.

$L^p(\Omega, \omega)$ is the space of measurable $f$ in $\Omega$, such that
\[
\|f\|_{L^p(\Omega, \omega)} = \left( \int_\Omega |f(x)|^p \omega(x)dx \right)^{\frac{1}{p}} < +\infty.
\]

$L^\infty(\Omega, \omega)$ is the space of measurable $f$ in $\Omega$, such that
\[
\|f\|_{L^\infty(\Omega, \omega)} = \inf \{ a \geq 0 : \omega(\{ x \in \Omega : |f(x)| > a \}) = 0 \} < +\infty.
\]

$Lip(\overline{\Omega})$ denotes the class of Lipschitz functions in $\overline{\Omega}$. $Lip_0(\Omega)$ denotes the class of functions $f \in Lip(\overline{\Omega})$ with compact support contained in $\Omega$. If $f \in Lip(\overline{\Omega})$, we can define the norm
\[
\|f\|_{H^{1,p}(\Omega, \omega)} = \|f\|_{L^p(\Omega, \omega)} + \|\nabla f\|_{L^p(\Omega, \omega)}.
\]

$H^{1,p}(\Omega, \omega)$ denotes the closure of $Lip(\overline{\Omega})$ under the norm $(\ref{eq:lipnorm})$. We say that $f \in H^{1,p}_{\text{loc}}(\Omega, \omega)$ if $f \in H^{1,p}(\Omega', \omega)$ for every $\Omega' \subset \subset \Omega$. $H^{1,p}_0(\Omega, \omega)$ denotes the closure of $Lip_0(\Omega)$ under the norm $(\ref{eq:lipnorm})$.

Now we introduce four kinds of weighted Morrey spaces in $\mathbb{R}^n$. Let $p \geq 1$, $\lambda \in R$, we have

Definition 2.2 Let $\|f\|_{L^p, \lambda(\mathbb{R}^n, \omega)} = \sup_{r > 0} \frac{r^{-\lambda}}{\omega(B_r(x))} \int_{B_r(x)} |f(y)|^p \omega(y)dy$. We set
\[
L^{p, \lambda}(\mathbb{R}^n, \omega) = \left\{ f \in L^p(\mathbb{R}^n, \omega) : \|f\|_{L^p, \lambda(\mathbb{R}^n, \omega)} < +\infty \right\}.
\]

Definition 2.3 Let $\|f\|_{\tilde{L}^p, \lambda(\mathbb{R}^n, \omega)} = \sup_{r > 0} \frac{\omega(B_r(x))^{-\lambda}}{\int_{B_r(x)} |f(y)|^p \omega(y)dy}$. We set
\[
\tilde{L}^{p, \lambda}(\mathbb{R}^n, \omega) = \left\{ f \in L^p(\mathbb{R}^n, \omega) : \|f\|_{\tilde{L}^p, \lambda(\mathbb{R}^n, \omega)} < +\infty \right\}.
\]

Definition 2.4 Let $\|f\|_{M^p, \lambda(\mathbb{R}^n, \omega)} = \sup_{x \in \mathbb{R}^n} \int_0^{+\infty} r^{-\lambda-1} \left( \int_{B_r(x)} |f(y)|^p \omega(y)dy \right) dr$. We set
\[
M^{p, \lambda}(\mathbb{R}^n, \omega) = \left\{ f \in L^p(\mathbb{R}^n, \omega) : \|f\|_{M^p, \lambda(\mathbb{R}^n, \omega)} < +\infty \right\}.
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