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HARMONIC MAPS AND CRITICAL POINTS OF PENALIZED ENERGY

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Abstract We discuss a sequence solutions $u_\varepsilon$for the E-L equations of the penalized energy defined by Chen-Struwe. We show that the blow-up set of u_ε is a H^{m-2} -rectifiable set and its weak limit satisfies a blow-up formula. Consequently, the weak limit will be a stationary harmonic map if and only if the blow-up set is stationary.
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1. Introduction

Let M, N be smooth compact Riemannian manifolds without boundary, and let $m = \dim M$. By Nash's embedding theorem , N can be viewed as a submanifold of R^k. Suppose that $u : M \to N$ is a map. We consider the penalized energy, which is defined by Chen-Struwe in [1]

$$I_\varepsilon(u) = \int_M \left(\frac{1}{2} |\nabla u|^2 + \frac{F(u)}{\varepsilon^2} \right) dV, \tag{1}$$

where $|\nabla u|^2 = g^{\alpha\beta} \frac{\partial u^i}{\partial x_\alpha} \frac{\partial u^i}{\partial x_\beta}$, $dV = \sqrt{\det(g_{\alpha\beta})} dx_1 \cdots dx_m$ in local coordinate, and $(g_{\alpha\beta})$ is the metric of M, $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$. Here and in the following a summation convention is used. $F(u)$ in (**??**) is a smooth functional of u such that

$$\begin{aligned} F(u) &= \mathrm{dist}^2(p, N), \quad \text{if } \mathrm{dist}(p, N) \le \delta, \\ &= 4\delta^2, \qquad\qquad\quad \text{if } \mathrm{dist}(p, N) \ge 2\delta, \end{aligned}$$

where δ is chosen so that $\mathrm{dist}^2(p, N)$ is smooth for $p \in \{p : \mathrm{dist}(p, N) \le 2\delta\}$. Guided by Chen-Lin in [2], we know that $I_\varepsilon(u)$ is unconstrained variational integral, which will facilitate our study of nonlinear and nonconvex constrained problems.

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The Eular-Lagrange equations for $I_\varepsilon(u)$ are

$$-\triangle_M u + \frac{1}{\varepsilon^2} f(u) = 0, \quad \text{in } M \tag{2}$$

where $f(u) = \text{grad } F(u)$. By classic elliptic theory, for any $\varepsilon > 0$, there exists a smooth solution u_ε of (**??**). If $I_\varepsilon(u_\varepsilon) < \Lambda$ for any $\varepsilon > 0$, then there exists a subsequence if needed such that $u_\varepsilon \rightharpoonup u$ weakly in $H^1_{loc}(M,N)$ as $\varepsilon \to 0$, where u is a weakly harmonic map. But we can't know whether u is a stationary harmonic map. Of course, we can't know whether this subsequence $\{u_\varepsilon\}$ converges strongly to u.

The strong convergence of $\{u_\varepsilon\}$ has been partially discussed in [3]. They proved that $u_\varepsilon \to u$ strongly in $H^1_{loc}(M,N)$ if there is no smooth nonconstant harmonic sphere from S^2 into N and consequently u is a stationary harmonic map.

However, the well-known theorem of Sacks-Uhlenbeck[4] guaranttees the existence of harmonic S^2.

Hence we take another way to discuss the strong convergence of $\{u_\varepsilon\}$. Let $\{u_\varepsilon\}$ be a sequence of smooth solutions of (**??**) with $I_\varepsilon(u_\varepsilon) \le \Lambda$. We define its blow-up set that

$$\Sigma = \bigcap_{r>0} \left\{ x \in M \,|\, \liminf_{\varepsilon \to 0} r^{2-m} \int_{B_r(x)} e(u_\varepsilon) dV \ge \varepsilon_0^2 \right\}$$

where $e(u_\varepsilon) = \frac{1}{2}|Du_\varepsilon|^2 + \frac{F(u_\varepsilon)}{\varepsilon^2}$ and ε_0 is a suitable positive constant. Assume that $u_\varepsilon \rightharpoonup u$ weakly in $H^1_{loc}(M,N)$, and $\mu_\varepsilon = e(u_\varepsilon)dx \rightharpoonup \mu = \frac{1}{2}|Du|^2 dx + \nu$ in the sense of measures as $\varepsilon \to 0$. Then our main results are

Theorem 1 *Σ is a H^{m-2} -rectifiable set. That is, ν is a H^{m-2} -rectifiable measure.*

Theorem 2 *Let $U \subset M$ be an open set and let ξ be a C^1 vector field with compact support in U. Then u satisfies the following blow-up formula*

$$\int_\Sigma \text{div}_\Sigma(\xi)\nu + \int_M \left(\frac{1}{2}|Du|^2 \text{div}\xi - \left\langle du(\nabla_\alpha \xi), du(\frac{\partial}{\partial x^\alpha}) \right\rangle \right) dV = 0. \tag{3}$$

Corollary 3 *u is a stationary harmonic map if and only if Σ is stationary.*

The motivation for our theorems comes from the work on stationary harmonic map by F.H.Lin [5] and J.Y. Li & G. Tian [6]. They proved that a sequence of stationary weakly harmonic maps has a rectifiable blow-up set and its weak limit satisfies a blow-up formula.

For simplicity, we shall present our proofs in the case where M is the unit ball in R^m. The general cases can be done in the same manner. Here we shall consider the weak solutions of

$$-\triangle u + \frac{1}{\varepsilon^2} f(u) = 0, \text{ in } B_1. \tag{4}$$