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STABILITY AND REGULARITY OF SUITABLY WEAK SOLUTIONS OF n-DIMENSIONAL MAGNETOHYDRODYNAMICS EQUATIONS

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Dedicated to the 80th birthday of Professor Zhou Yulin

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Abstract In this paper, it is shown that the weak solutions of magnetohydrodynamics equations in spaces $L^q(R^+;\ L^p(R^n))$ are stable and regular.

Key Words Stability; regularity; magnetohydrodynamics equations.
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1. Introduction

The Magnetohydrodynamics equations and the incompressible Navier-Stokes equations play very important roles in nonlinear partial differential equations with dissipation. In this work, the author is concerned with stability and regularity of the following $n(\geq 3)$-dimensional Magnetohydrodynamics equations

$$u_t + (u\cdot\nabla)u - (A\cdot\nabla)A - \triangle u + \nabla p = 0, \qquad \nabla\cdot u = 0, \qquad \text{in } \mathbb{R}^n\times\mathbb{R}^+, \tag{1}$$
$$u(x,0) = u_0(x), \qquad \nabla\cdot u_0 = 0, \qquad \text{in } \mathbb{R}^n, \tag{2}$$
$$A_t + (u\cdot\nabla)A - (A\cdot\nabla)u - \triangle A = 0, \qquad \nabla\cdot A = 0, \qquad \text{in } \mathbb{R}^n\times\mathbb{R}^+, \tag{3}$$
$$A(x,0) = A_0(x), \qquad \nabla\cdot A_0 = 0, \qquad \text{in } \mathbb{R}^n, \tag{4}$$

where $u(x,t) = (u_1(x,t), u_2(x,t),\cdots,u_n(x,t))$ and $A(x,t) = (A_1(x,t), A_2(x,t),\cdots, A_n(x,t))$ are unknown vector-valued functions; and $p = p(x,t)$ is a real-valued function, representing pressure. In addition

$$\nabla\cdot u = \sum_{j=1}^n \frac{\partial u_j}{\partial x_j}, \qquad (u\cdot\nabla)u = \sum_{j=1}^n u_j\frac{\partial u}{\partial x_j}, \qquad \triangle u = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}.$$

Suppose that the weak solutions $(u,\ A,\ p)$ of the Cauchy problems (1)-(4) satisfy

$$\lim_{|x|\to\infty} \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}(u(x,t),\ A(x,t),\ p(x,t)) = 0, \tag{5}$$

where $\alpha_1 \geq 0, \alpha_2 \geq 0, \cdots, \alpha_n \geq 0$ are integers. Notice that if $A_0 \equiv 0$, then a simple argument shows that $A \equiv 0$, and (1) reduces to the following incompressible Navier-Stokes equations

$$u_t + (u \cdot \nabla)u - \triangle u + \nabla p = 0, \qquad \nabla \cdot u = 0 \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^+. \tag{6}$$

Let the initial data $(u_0,\ A_0) \in L^2(\mathbb{R}^n)$. Then the problem (1)-(4) admit a global weak solution $(u,\ A) \in L^\infty(0,T; L^2(\mathbb{R}^n)) \bigcap L^2(0,T; H^1(\mathbb{R}^n))$ and $p \in L^\infty(0,T; L^1(\mathbb{R}^n)) \bigcap L^2(0,T; W^{1,1}(\mathbb{R}^n))$, where $T > 0$ is any constant. This is a well known result, see [1]. However, the weak solutions are not unique in general. On the other hand, under the additional restrictions on the weak solutions: $(u,\ A) \in L^q(\mathbb{R}^+; L^p(\mathbb{R}^n))$, where $n/p + 2/q = 1$ and $p > n \geq 3$, then the global weak solution is unique. Our calculations in this paper show that if such solutions exist, then they are actually very strong. They are almost equivalent to smooth solutions for all $n \geq 3$. Let $(u,\ A,\ p)$ and $(v,\ B,\ q)$ be the solutions of problem (1)-(4) corresponding to the initial data $(u_0,\ A_0)$ and $(v_0,\ B_0)$, respectively, such that the above assumptions are satisfied. Let $(w,\ E,\ \pi) = (u - v,\ A - B,\ p - q)$. Then they satisfy the equations

$$w_t + [(w \cdot \nabla)u + (v \cdot \nabla)w] - [(E \cdot \nabla)A + (B \cdot \nabla)E] - \triangle w + \nabla \pi = 0, \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \tag{7}$$

$$E_t + [(w \cdot \nabla)A + (v \cdot \nabla)E] - [(E \cdot \nabla)u + (B \cdot \nabla)w] - \triangle E = 0, \qquad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \tag{8}$$

where $\nabla \cdot w = \nabla \cdot E = 0$ in $\mathbb{R}^n \times \mathbb{R}^+$, together with the initial conditions

$$w(x,0) = w_0(x) \;=\; u_0(x) - v_0(x), \qquad \nabla \cdot w_0 = 0, \qquad \text{in } \mathbb{R}^n, \tag{9}$$

$$E(x,0) = E_0(x) \;=\; A_0(x) - B_0(x), \qquad \nabla \cdot E_0 = 0, \qquad \text{in } \mathbb{R}^n. \tag{10}$$

Notations Denote by C any positive, time-independent constant, which may be different from one place to another place, and may depend on the initial data (u_0, A_0). Denote by $L^p(\mathbb{R}^n)$ and $H^m(\mathbb{R}^n)$ the usual functional spaces, where $p \in [1, +\infty]$ and $m \geq 1$. Let $f = (f_1, f_2, \cdots, f_n) \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n))$ and $g = (g_1, g_2, \cdots, g_n) \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n))$. Define

$$\begin{aligned}
&x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n, \qquad |x|^2 = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2, \\
&|f|^2 = |f_1|^2 + |f_2|^2 + \cdots + |f_n|^2, \qquad |g|^2 = |g_1|^2 + |g_2|^2 + \cdots + |g_n|^2, \\
&|(f,\ g)|^2 = |f|^2 + |g|^2, \\
&|(f,\ g)(\cdot,t)\|^2 = \|(f,\ g)(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left[|f(x,t)|^2 + |g(x,t)|^2\right] dx, \\
&f \cdot g = f_1 g_1 + f_2 g_2 + \cdots + f_n g_n, \\
&\nabla f \cdot \nabla g = \sum_{i=1}^n \nabla f_i \cdot \nabla g_i = \sum_{i=1}^n \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} \frac{\partial g_i}{\partial x_j}, \text{ if } f, g \in L^2(0,T; H^1(\mathbb{R}^n)).
\end{aligned}$$

Let $\varphi(x) \in L^1(\mathbb{R}^n) \bigcap L^2(\mathbb{R}^n)$, define its Fourier transform and inverse Fourier transform by