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A REMARK ON THE REGULARITY OF SOLUTIONS TO THE NAVIER–STOKES EQUATIONS*

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Abstract In this note we shall give a simple proof of a result in [1] which gives a sufficient condition for the regularity of solutions to the Navier–Stokes equation in $\mathbb{R}^n$ based on estimates on the vorticity.

Key Words Regularity, Cauchy problem, Navier–Stokes equation, admissible triplet, time–space estimates.

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1. Introduction and the Main Results

In this note we are concerned with the following Cauchy problem in $\mathbb{R}^n \times (0,T)$

$$\partial_t v - \triangle v + (v\cdot\nabla)v + \nabla P = 0, \tag{1.1}$$

$$\mathrm{div}\ v = 0, \tag{1.2}$$

$$v(0) = v_0(x), \tag{1.3}$$

where $v(t) = v(t,x) = (v_1(t,x), v_2(t,x), \cdots, v_n(t,x))$, is the velocity field, P is the pressure.

Definition 1.1 *A vector field $v \in L^\infty((0,T); L^2(\mathbb{R}^n)) \cap L^2((0,T); \dot{H}^1(\mathbb{R}^n))$ is called the Leray–Hopf weak solution if*

$$\int_0^T \int_{\mathbb{R}^n} [v\cdot\varphi_t + (v\cdot\nabla)\varphi\cdot v + v\cdot\triangle\varphi]dxdt = 0,$$
$$for \quad \forall\varphi \in [\mathcal{C}_0^\infty(\mathbb{R}^n\times(0,T)]^n, \quad with \quad div\ \varphi = 0, \tag{1.4}$$

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and

$$div\ v = 0, \tag{1.5}$$

in the distributional sense.

For $v_0 \in L^2(\mathbb{R}^n)$ with div $v_0 = 0$, the global existence of weak solution was established by Leray and Hopf in [2] and [3]. It is still unknown whether the Leray–Hopf weak solution to the Navier–Stokes equations is unique. As for the strong solution or $L^q(I; L^p)$–solutions, it is well known that for $v_0 \in H^1(\mathbb{R}^n)$ $(n \le 4)$ with div $v_0 = 0$ or $v_0 \in L^r(\mathbb{R}^n)$ $(r \ge n)$ with div $v_0 = 0$ in distributional sense, then there exists a local unique strong solution $v \in \mathcal{C}([0,T); H^1(\mathbb{R}^n))$ $(n \le 4)$ or $L^q(I; L^p)$–solution for any space dimensions, where the maximal time existence T_* depends on the initial data $\|v_0 :\ H^1(\mathbb{R}^n)\|$ $(n \le 4)$ or $\|v_0\|_r$ in the subcritical case $r > n$ and depends on v_0 itself in the critical case $r = n$, for details see [4–13] and [14]. As an immediate consequence of regularity of analytic semigroup which is generated by the Stokes operator, one easily sees that the strong solution $(n \le 4)$ and the $L^q(I; L^p)$–solution belong to the $\mathcal{C}((0,T); \mathcal{C}^\infty(\mathbb{R}^n))$, see [9] and [13]. The global in time existence of strong solution or $L^q(I; L^p)$–solution is an outstanding open problem. Many authors have deduced the sufficient conditions under which the Leray–Hopf weak solution agrees with the smooth solution. In this direction, there is a classical result due to Serrin [12], which states that if a Leray–Hopf weak solution belongs to $L^q(I; L^p(\mathbb{R}^3))$, $\frac{2}{q} + \frac{3}{p} < 1$ and $q < \infty$, then v becomes the smooth solution. Later, Fabes, Jone and Riviere in [4] extend the above criterion to the case $\frac{2}{q} + \frac{3}{p} = 1$. The case $q = \infty$, $p = 3$ in Serrin's conditions, regularity and uniqueness of the solution to the Navier–Stokes equations was established in [13]. For general space dimension case $(\frac{2}{q} + \frac{n}{p} \le 1)$ has been studied by many authors, see [8,9] and [13] and references therein.

Recently, Beirão da Veiga [1] obtained a sufficient condition for regularity using the vorticity $w =$ curl v, rather than the velocity v, his results can be stated as follows:

Theorem 1.1 *Let $v_0 \in L^2(\mathbb{R}^3)$ with div v_0=0 and $w_0 =$ curl $v_0 \in L^2(\mathbb{R}^3)$. If the Leray–Hopf weak solution v satisfies $w =$ curl $v \in L^q(I; L^p(\mathbb{R}^3))$ with $\frac{2}{q} + \frac{3}{p} \le 2$, $1 < q < \infty$, then v becomes the classical solution on $I = (0,T)$.*

In [15] Dongho Chae & Hi–Jun Choe extended the results of [1] as:

Theorem 1.2 *Let $v_0 \in L^2(\mathbb{R}^3)$ with div v_0=0 and $\omega_0 =$ curl $v_0 \in L^2(\mathbb{R}^3)$. Let v be the Leray–Hopf weak solution to (1.1), $w =$ curl v. Assume that $\tilde{\omega} \in L^q(I; L^p(\mathbb{R}^3))$ with $\frac{2}{q} + \frac{3}{p} \le 2$, $1 < q < \infty$, where*

$$\tilde{\omega} = \omega_1 e_1 + \omega_2 e_2, \quad e_1 = (1,0,0), \quad e_2 = (0,1,0). \tag{1.6}$$

Then v becomes the classical solution on $I = (0,T)$.

In (1.6) $\omega_1 e_1$ or $\omega_2 e_2$ can be replaced by $\omega_3 e_3$, which means that the regularity of the solution of (1.1) depends on two components of the vorticity field.

In this note, we shall give a simple proof of Theorem 1.1 and its generalization in higher dimensions.