J. Partial Diff. Eqs. **16**(2003), 37–48
©International Academic Publishers Vol.16, No.1

TWO DIMENSIONAL INTERFACE PROBLEMS FOR ELLIPTIC EQUATIONS*

Ying Lung-an
(Key Laboratory of Pure and Applied Mathematics
School of Mathematical Sciences
Peking University
Beijing, 100871, China)

Dedicated to the 80th birthday of Professor Zhou Yulin

(Received Feb. 28, 2002)

Abstract We study the structure of solutions to the interface problems for second order quasi-linear elliptic partial differential equations in two dimensional space. We prove that each weak solution can be decomposed into two parts near singular points, a finite sum of functions in the form of $cr^\alpha \log^m r\varphi(\theta)$ and a regular one w. The coefficients c and the $C^{1,\alpha}$ norm of w depend on the H^1-norm and the $C^{0,\alpha}$-norm of the solution, and the equation only.

Key Words Quasilinear elliptic equations; interface problems; weak solutions; singular points.

2000 MR Subject Classification 35J60, 35D05.

Chinese Library Classification O.175.25, O.175.29.

1. Introduction

We study the structure of the solutions to the equation

$$\frac{\partial}{\partial x_j}\left(a_{ij}(x,u)\frac{\partial u}{\partial x_i}\right) = \frac{\partial f_i}{\partial x_i}, \quad x \in \Omega_0, \tag{1}$$

where $\Omega_0 \subset \mathbb{R}^2$ and a_{ij}, f_i are discontinuous functions, $i,j = 1,2$. The summation convention is assumed here. It is known that if u is a weak solution in $H^1(\Omega)$ then $u \in C^{0,\alpha}(\Omega)$ with a certain $\alpha \in (0,1)$. Moreover, if a_{ij}, f_i are piecewise smooth, then the solutions possess some structure near the discontinuous points of the coefficients. This kind of interface problems has been studied by a number of authors [1–8]. In [8] we proved that each weak solution to (1) can be decomposed into two parts near a singular point, a singular part and a regular part. The singular part is a finite sum of particular solutions with the form of $r^\alpha\varphi(\theta)$, or $r^\alpha \log^m r\varphi(\theta)$, where r is the distance to the singular point, and θ is the polar angle, and the regular part is bounded with respect to a norm which is slightly weaker than the H^2 norm, multiplied by a factor $\frac{1}{(|\log r|+1)^M}$.

*This work was supported by the Major State Research Program of China G1999032803 and the Research Fund for the Doctorial Program of Higher Education.

The result in [8] does not imply the boundedness of the derivatives of the regular part. The aim of this paper is to study the $C^{1,\alpha}$ norm estimate of the regular part. Our result is optimal here, that is, the regularity of the regular part of a weak solution is the same as the regularity of those solutions for the equations with smooth coefficients a_{ij}.

Let us present a statement of the problem and the main result. Let Ω_0 be a polygonal domain. We assume that Ω_0 is decomposed into a finite number of polygonal sub-domains $\Omega^{(k)}$, such that $\cup\overline{\Omega^{(k)}} = \overline{\Omega_0}$, and a_{ij} are sufficiently smooth on $\overline{\Omega^{(k)}} \times \mathbb{R}$. Moreover, we assume that a_{ij} satisfy the following elliptic condition:

$$a_{ij}(x,u)\xi_i\xi_j \geq \kappa|\xi|^2, \forall \xi \in \mathbb{R}^2,$$

for all $(x,u) \in (\Omega_0 \times \mathbb{R})$, where κ is a positive number. We also assume that $f_i \in C^{0,\alpha}(\Omega^{(k)})$ with $\alpha \in (0,1)$. For simplicity we impose the Dirichlet boundary condition,

$$u|_{x\in\partial\Omega_0} = 0 \tag{2}$$

on (1), where $\partial\Omega_0$ is the boundary.

The following points will be generally known as singular points: the cross points of interfaces, the turning points of interfaces, the cross points of interfaces with the boundary $\partial\Omega_0$, and the points on $\partial\Omega_0$ with interior angles greater than π. Let Σ be the set of singular points. We assume that Σ is a finite set. The problem (1) (2) admits a solution $u \in H_0^1(\Omega_0)$ (see [9–11]), and it is easy to prove that for each sub-domain $\Omega^{(k)}$, $u \in C^{1,\alpha}_{\text{loc}}(\Omega^{(k)}\backslash\Sigma)$. Thus the problem is the behavior of u near the singular points.

Let x_0 be a singular point. We construct local polar coordinates (r,θ) with the origin x_0. Let $s(x_0,\rho) \subset \Omega_0$ be a disc with center x_0 and radius ρ, such that x_0 is the only singular point on the disc. The subsets $s(x_0,\rho) \cap \Omega^{(k)}$ are thus some sectors, denoted by S_m. The main result of this paper is the following:

Theorem 1.1 *Let u be a weak solution to (1) (2) and $u \in H^1(\Omega_0)\bigcap C^{0,\bar{\delta}}(\Omega_0)$, $\bar{\delta} \in (0,1)$. Then there is an integer N and a constant $\alpha_0 \in (0,\bar{\delta}]$, such that if $0 < \alpha < \alpha_0$ then $u = \sum_{n=1}^N u_n + w$ on $s(x_0,\rho)$, where*

$$u_n = c_n r^{\alpha_n} \log^{m_n} r \varphi_n(\theta), \tag{3}$$

$$\sum_m \|Dw\|_{C^{0,\alpha}(S_m)} + \sum_n |c_n| \leq C, \tag{4}$$

where m_n are non-negative integers, and φ_n are continuous, periodic, and piecewise infinitely differentiable functions, which depend only on $a_{ij}(x_0,u(x_0))$ and n; and C depends only on a_{ij}, $\|u\|_{H^1(\Omega_0)}$, $\|u\|_{C^{0,\bar{\delta}}(\Omega_0)}$, and $\|f_i\|_{C^{0,\alpha}(\Omega^{(k)})}$.

We will study homogeneous equations with constant coefficients in the next section, and nonhomogeneous equations with constant coefficients in Section 3, then prove the main theorem in Section 4. In what follows we assume that the singular point is an interior point. For those singular points on the boundary the argument is analogous. Without loss of generality we assume throughout this paper that the radius $\rho = 1$, the singular point $x_0 = 0$, and C is a generic constant possessing the above property.