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THE $W_\infty^{1,0}$ ESTIMATE FOR STRONG SOLUTIONS OF FULLY NONLINEAR PARABOLIC PDE'S

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Abstract In this paper, we deal with the following fully nonlinear partial differential equation of parabolic type

$$F(x, t, u, Du, D^2u) + u_t = 0 \quad \text{in } Q_T$$

with the third boundary value conditions. We prove that under some structure conditions on F, the $W_p^{2,1}$ $(p > 3N + 1)$ strong solutions for the problem have the $W_\infty^{1,0}$ a priori estimates.

Key Words Fully nonlinear parabolic equations; third boundary value conditions; strong solutions; $W_\infty^{1,0}$ estimates.
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1. Introduction

In this paper we consider the following fully nonlinear partial differential equations of parabolic type

$$F(x, t, u, Du, D^2u) + u_t = 0 \qquad \text{in } Q_T \tag{1.1}$$

with the initial and boundary value conditions:

$$\frac{\partial u(x,t)}{\partial n} = b(x,t)u(x,t) + c(x,t) \qquad \text{on } \partial\Omega \times [0, T] \tag{1.2}$$

$$u(x, 0) = \varphi(x) \qquad \text{in } \Omega \tag{1.3}$$

where $Q_T = \Omega \times (0, T] \subset R^N \times R^+, \partial\Omega \in C^2, Du, D^2u$ represent the gradient and Hessian of $u(x, t)$ in x respectively, $u_t = \dfrac{\partial u(x,t)}{\partial t}$ and $b(x, t), c(x, t) \in C^{1,1}(\bar{Q}_T), \varphi(x) \in C(\bar{\Omega}), n$ is the outward unit normal to $\partial\Omega, F$ is a continuous function on $\bar{Q}_T \times R \times R^N \times S^N$ and S^N is the space of $N \times N$ symmetric matrices.

In [1] and [2] the authors have proved the existence and uniqueness of classical solution for (1.1) under the first initial and boundary value conditions, and in [3] , Professor Dong has proved the existence and uniqueness of classical solution for (1.1) with the fully nonlinear boundary value conditions. All the existence results mentioned above need the concavity condition on F in $X \in S^N$. When F does not satisfy the concavity condition, Professor Dong has proved in [4] the existence and uniqueness of $W^{1,0}_\infty$ viscosity solution for (1.1) with the first initial and boundary value conditions. The main methods used in that paper are $W^{1,0}_\infty$ estimates for strong solutions and m-accretive operator approximation of the problem. For (1.1) (1.2) and (1.3) type problem, the authors in [5–7] have proved the comparison theorem for the equation (1.1) and proved the existence of viscosity solutions for the problem (1.1)(1.2) (1.3) under the assumption that the problem has a viscosity supersolution and a viscosity subsolution. But they did not indicate the existence of viscosity supersolution and subsolution.

In this paper, follow the idea in [8] and [4] , we mainly prove the $W^{1,0}_\infty$ estimate for the problem (1.1) (1.2) (1.3) under some structure conditions on F. With this basic estimate, we can use the m-accretive operator approximation to construct the $W^{1,0}_\infty$ viscosity solution for the problem (1.1) -(1.3), and prove the existence of $W^{1,0}_\infty$ viscosity solutions.

2. The $W^{1,0}_\infty$ a priori estimate

In this section, we consider the initial and boundary value problem (1.1)-(1.3). We make the following hypotheses:

F1 : $\Lambda^{-1}|\xi|^2 \le -\sum_{i,j=1}^N F_{r_{ij}}\xi_i\xi_j \le \Lambda|\xi|^2, \quad \forall \xi \in R^N,$

F2 : $|F(x,t,u,p,0)| \le \Lambda(1+|p|^2),$

F3 : $|F| + |F_u| + \sum_{i=1}^N |F_{p_i}|(1+|p|) + \sum_{i=1}^N |F_{x_i}|(1+|p|)^{-1}$
$\qquad \le \mu_0(|u|)(1+|p|^2+|r|),$

F4 : $F_u(x,t,u,p,r) \ge \mu_1 > 0,$

B1 : $\partial\Omega \in C^2, n$ the outward unit normal to $\partial\Omega,$

B2 : $b(x,t), c(x,t) \in C^{1,1}(\bar{Q}_T),$ and $b(x,t) \le b_0 < 0,$

where Λ, μ_1 and b_0 are constants, $\mu_0(|z|)$ is a nondecreasing function of $|z|$.

Lemma 2.1 *If $u \in W^{2,1}_p(Q_T)(p > N+2)$ is a strong solution for* (1.1)-(1.3). *Assume that F1, F2, F4, and B1, B2 holds, then $|u|_{L^\infty} \le M_0$, where M_0 depends only on Λ, μ_1, b_0.*

Lemma 2.2 *Suppose $u \in W^{2,1}_p(Q_T)(p > N+2)$ is a strong solution for* (1.1)-(1.3), $\max_{Q_T}|u| \le M_0$, *and F1, F2, B1, B2 hold, then there are constants α, M_α depending only on N, Λ, M_0, $\mu_1, Q_T, \partial\Omega$, such that:*

$$\|u\|_{C^{\alpha,\alpha/2}(\bar{Q}_T)} \le M_\alpha.$$

The above two Lemmas can be proved in a similar way as in [2, 3], we omit the proofs here.