J. Partial Diff. Eqs. **15**(2002), 46–68
©International Academic Publishers Vol.15, No.1

CAUCHY PROBLEM FOR GENERAL FIRST ORDER INHOMOGENEOUS QUASILINEAR HYPERBOLIC SYSTEMS

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(Received Sept. 10, 2001)

Abstract In this paper, we consider Cauchy problem for general first order inhomogeneous quasilinear strictly hyperbolic systems. Under the matching condition, we first give an estimate on inhomogeneous terms. By this estimate, we obtain the asymptotic behaviour for the life-span of C^1 solutions with "slowly" decaying and small initial data and prove that the formation of singularity is due to the envelope of characteristics of the same family.

Key Words Quasilinear hyperbolic system; matching condition; life-span; weak linear degeneracy.

2000 MR Subject Classification 35L45, 35L60.

Chinese Library Classification O175.22, O175.27.

1. Introduction and Main Results

Consider the following first order inhomogeneous quasilinear system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u), \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of $(t, x), A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)(i, j = 1, \cdots, n)$ and $F(u) = (f_1(u), \cdots, f_n(u))^T$ is a vector function of u with suitably smooth elements $f_i(u)(i = 1, \cdots, n)$.

Suppose that the system (1.1) is strictly hperbolic in a neighbourhood of $u = 0$, namely, for any given u in this domain, $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{r1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvecdtor corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \text{ (resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)) \tag{1.3}$$

We have

$$\det|l_{ij}(u)| \neq 0 \text{ (resp. } \det|r_{ij}(u)| \neq 0). \tag{1.4}$$

All $\lambda_i(u), l_{ij}(u)$ and $r_{ij}(u)(i,j = 1,\cdots,n)$ have the same regularity as $a_{ij}(u)(i,j = 1,\cdots,n)$. Without loss of generality, we may suppose that

$$l_i(u)r_j(u) \equiv \delta_{ij}, \quad i,j = 1,\cdots,n \tag{1.5}$$

and

$$r_i^T(u)r_i(u) \equiv 1, \quad i = 1,\cdots,n, \tag{1.6}$$

where δ_{ij} stands for Kronecker's symbol.

For the following initial data

$$t = 0: \quad u = \varphi(x), \tag{1.7}$$

where $\varphi(x)$ is a "small" C^1 vector function of x with certain decay properties as $|x| \to \infty$, Li et al.[1,2] presented a complete result on the global existence and the blow-up phenomenon of C^1 solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.7) in the case $F(u) \equiv 0$. In the case that $F(u)$ satisfies the so-called matching condition, Koing [3] gave a quite complete result for the global existence and the breakdown of C^1 solution $u = u(t,x)$ to Cauchy problem (1.1) and (1.7). Kiong [4] also proved that the results given in [2] on the breakdown of C^1 solution are still valid for "slow" decaying initial data. In this paper, in the case that the inhomogeneous term satisfies the matching condition, we will prove that the system (1.1) has the same result as in the homogeneous case. We will first give an estimate on the inhomogeneous term under the matching condition. By this estimate, the corresponding proof given in [3] can be simplified. On the other hand, we generalize the results in [3] on the breakdown of C^1 solution for "slow" decaying initial data.

For the completeness of statement, we first recall the concepts of the weak linear degeneracy (see [5] or [1]) and the matching condition (see [6] or [3]).

Definition 1.1 *The i-th characteristic $\lambda_i(u)$ is weakly linearly degenerate if along the i-th characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$, defined by*

$$\begin{cases} \dfrac{du}{ds} = r_i(u), \\ s = 0: \quad u = 0, \end{cases} \tag{1.8}$$

we have

$$\nabla\lambda_i(u)r_i(u) \equiv 0, \ \forall|u| \text{ small}, \quad \text{namely}, \quad \lambda_i(u^{(i)}(s) \equiv \lambda_i(0), \quad \forall|s| \text{ small} \tag{1.9}$$

If all characteristics are wakly linearly degenerate, the system (1.1) is said to be weakly linearly degenerate.