J. Partial Diff. Eqs. **13**(2000), 243–252
Vol.13, No.3

QUENCHING VERSUS BLOW-UP

Deng Keng and Zhao Chenglin
(Department of Mathematics, University of Southwestern Louisiana, Lafayette, Louisiana 70504)
(Received Aug. 27, 1999)

Abstract This paper is concerned with the semilinear heat equation $u_t = \Delta u - u^{-q}$ in $\Omega \times (0,T)$ under the nonlinear boundary condition $\dfrac{\partial u}{\partial \nu} = u^p$ on $\partial\Omega \times (0,T)$. Criteria for finite time quenching and blow-up are established, quenching and blow-up sets are discussed, and the rates of quenching and blow-up are obtained.
Key Words Reaction-diffusion equation; finite time quenching and blow-up; quenching and blow-up sets; quenching and blow-up rates.
1991 MR Subject Classification 35B40, 35K55, 35K60.
Chinese Library Classification O175.26, O175.8, O175.29.

1. Introduction

In this paper, we consider the following initial-boundary value problem:

$$\begin{aligned} &u_t = \Delta u - u^{-q}, && x \in \Omega,\ t > 0 \\ &\frac{\partial u}{\partial \nu} = u^p, && x \in \partial\Omega,\ t > 0 \\ &u(x,0) = u_0(x), && x \in \Omega \end{aligned} \tag{1.1}$$

where $0 < p, q < \infty, \Omega$ is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega, \nu$ is the outward normal, $0 \le u_0(x) \le M$, and $\dfrac{\partial u_0}{\partial \nu} = u_0^p$ on $\partial\Omega$.

Physically, (1.1) can be treated as a heat conduction model that incorporates the effects of reaction and nonlinear influx. Mathematically, (1.1) is a combination of the following two problems:

$$\begin{aligned} &u_t = \Delta u - u^{-q}, && x \in \Omega,\ t > 0 \\ &\frac{\partial u}{\partial \nu} = 0, && x \in \partial\Omega,\ t > 0 \\ &u(x,0) = u_0(x), && x \in \Omega \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} &u_t = \Delta u, && x \in \Omega,\ t > 0 \\ &\frac{\partial u}{\partial \nu} = u^p, && x \in \partial\Omega,\ t > 0 \\ &u(x,0) = u_0(x), && x \in \Omega \end{aligned} \tag{1.3}$$

As is well known, all solutions of the problem (1.2) quench (solutions reach zero) in finite time for $q > 0$ (cf. [1]) while every solution of the problem (1.3) blows up in finite time for $p > 1$ (cf. [2]). Moreover, for (1.2), the rate near the quenching time T is $(T-t)^{\frac{1}{q+1}}$ [3] while for (1.3), the rate near blow-up is $(T-t)^{-\frac{1}{2(p-1)}}$ and blow-up occurs only on the boundary [2,4]. From these facts, a natural question arises: What is the behavior of the solution of the problem (1.1)? Our main objective here is to answer the above question. Specifically, we will show whether the solution of (1.1) quenches or blows up in finite time depends upon certain conditions on the initial data. We will also characterize the quenching and blow-up sets, and show that the rate estimates near quenching and blow-up are the same as those for (1.2) and (1.3), respectively. This means that even with the presence of the nonlinear influx, the "small" solutions of (1.1) still behave like those of (1.2) whereas even with the introduction of the negative reaction, the "large" solutions of (1.1) still behave like those of (1.3).

The plan of the paper is as follows: In Section 2, we establish the criteria for quenching, while in Section 3, we establish the criteria for blow-up. In Section 4, we discuss the quenching and blow-up sets, and in Section 5, we derive estimates for the quenching and blow-up rates.

In the sequel, a solution of (1.1) is always understood in the classical sense.

2. Finite Time Quenching

In this section, we establish two results concerning the finite time quenching of solutions of the problem (1.1). For that purpose, we should always assume that $u_0(x) > 0$ for $x \in \bar{\Omega}$. With a monotonicity assumption, we first present the following result.

Theorem 2.1 *Suppose that* $\Delta u_0(x) - u_0^{-q}(x) \le 0$ *for* $x \in \Omega, p > 0$, *and* $q > 0$. *Then the solution of* (1.1) *quenches in finite time.*

Proof Note that the condition on the initial datum implies $u_t(x,t) \le 0$ for $x \in \bar{\Omega}$ and $t > 0$. Moreover, if we let $\gamma = \int_\Omega u_0^{-q}dx - \int_{\partial\Omega} u_0^p dx$, then $\gamma \ge 0$. Introduce $F(t) = \int_\Omega u(x,t)dx$, we find

$$F'(t) = \int_\Omega u_t(x,t)dx = \int_{\partial\Omega} u^p dx - \int_\Omega u^{-q}dx \le \int_{\partial\Omega} u_0^p dx - \int_\Omega u_0^{-q}dx = -\gamma$$

Thus

$$F(t) \le F(0) - \gamma t$$

which means that there exists a point $(x_0, t_0) \in \bar{\Omega} \times (0,\infty)$ such that $u(x_0,t_0) = 0$.

Next, we present another quenching result by replacing the monotonicity assumption with a relatively weak condition on $u_0(x)$.

Theorem 2.2 *Assume that* $u_0(x) \ll 1, p \ge 1$, *and* $q > 0$. *Then the solution of* (1.1) *quenches in finite time.*