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HEISENBERG'S INEQUALITY AND LOGARITHMIC HEISENBERG'S INEQUALITY FOR AMBIGUITY FUNCTION*

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Abstract In this article we discuss the relation between Heisenberg's inequality and logarithmic Heisenberg's (entropy) inequality for ambiguity function. After building up a Heisenberg's inequality, we obtain a connection of variance with entropy by variational method. Using classical Taylor's expansion, we prove that the equality in Heisenberg's inequality holds if and only if the entropy of $2k-1$ order is equal to $(2k-1)!$.

Key Words Heisenberg's inequality; ambiguity function; logarithmic Heisenberg's inequality; entropy.
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1. Introduction

The ambiguity function introduced by Woodward [1] is important in radar signal analysis. It is a function of two real variables, τ (the time) and ω (with $2\pi\omega$ being the frequency) and is defined as follows in terms of two given functions f and g of one variable

$$A_{f,g}(\tau,\omega)=\int f\Big(t-\frac{1}{2}\tau\Big)\overline{g\Big(t+\frac{1}{2}\tau\Big)}e^{-2\pi i\omega t}dt \tag{1.1}$$

where $f,g\in L^2(R)$, $\bar{g}$ is the conjugate complex function of g and $i=\sqrt{-1}$. If we define the Fourier tranсform $\hat{f}$ of a function f to be

$$\hat{f}=\int f(t)e^{-2\pi i\omega t}dt$$

then $A_{f,g}$ has another form:

$$A_{f,g}(\tau,\omega)=\int \hat{f}\Big(t+\frac{\omega}{2}\Big)\overline{\hat{g}\Big(t-\frac{\omega}{2}\Big)}e^{-2\pi it\tau}dt \tag{1.2}$$

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In radar engineering literature, $A_{f,g}$ is known as the cross ambiguity function of f and g, and $A_{f,f}$ truely the ambiguity function of f. Although we can not prove that $A_{f,f} \geq 0$ a.e., it also has a Heisenberg's inequality (See Folland, P.223, [2])

$$\inf_{a,b} \int (|\tau - a|^2 + |\omega - b|^2) A_{f,f}(\tau,\omega) d\tau d\omega \geq \frac{1}{2\pi} \tag{1.3}$$

if $\|f\|_2 = 1$. But for $|A_{f,g}|^2$, Heisenberg's inequality, which was neglected and hence has not been built up, is an important property of ambiguity functions (See Corollary 2.3 below).

Theorem 1.1 (Heisenberg's inequality) *If $f, g \in L^2(R)$, and $\|f\|_2 \|g\|_2 = 1$, then*

$$\inf_{a,b \in R} \int [(\tau - a)^2 + (\omega - b)^2] |A_{f,g}(\tau,\omega)|^2 d\tau d\omega \geq \frac{1}{\pi} \tag{1.4}$$

We have used the standard notations

$$\|f\|_p = \left(\int |f(t)|^p dt \right)^{1/p}, \quad \text{for} \quad 1 \leq p < \infty$$

and

$$\|f\|_\infty = \text{ess sup} |f(t)|, \quad \text{for } p = \infty$$

When $p = 2$, Parseval's relation is

$$\|\hat{f}\|_2 = \|f\|_2$$

and then Schwartz inequality yields the pointwise-bound

$$|A_{f,g}(\tau,\omega)| \leq \|f\|_2 \|g\|_2 = \|\hat{f}\|_2 \|\hat{g}\|_2 \tag{1.5}$$

We now introduce the following notation

$$I_{f,g}(p) = \int \int |A_{f,g}(\tau,\omega)|^p d\tau d\omega$$

When $p = 2$, by Parseval's formula, we have the identity

$$I_{f,g}(2) = \int |f(t)|^2 dt \int |g(t)|^2 dt \tag{1.6}$$

In signal analysis, Gaussian signals are of special use because their imput and output signals are of the same type. This can be shown from the formula (Th. 1.15, [3])

$$(e^{-\alpha\pi t^2})^\wedge(\xi) = \alpha^{-1/2} e^{-\alpha^{-1}\pi\xi^2} \tag{1.7}$$

for Re $\alpha > 0$. So we give some definitions relating to Gaussian functions.

Definition 1.1 *$f(t)$ is said to be a Gaussian if*

$$f(t) = e^{-\alpha\pi t^2 - \beta t - \gamma}$$