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# Large Time Behaviour of the Solution of a Nonlinear Diffusion Problem in Anthropology

Ján Eliaš[1,*], Danielle Hilhorst[2] and Masayasu Mimura[3,4]

[1] *Institute of Mathematics and Scientific Computing, University of Graz, Heinrichstrasse 36, Graz 8010, Austria.*
[2] *Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Université Paris-Saclay, Orsay 91405, France.*
[3] *Meiji Institute for Advanced Study of Mathematical Sciences, Meiji University, 4–21–1 Nakano, Nakano ku, Tokyo 164–8525, Japan.*
[4] *Department of Mathematical Engineering, Musashino University, 3–3–3 Ariake, Koto-ku, Tokyo 135–8181, Japan.*

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**Abstract.** In this article we consider a reaction-diffusion model for the spreading of farmers in Europe, which was occupied by hunter-gatherers; this process is known as the Neolithic agricultural revolution. The spreading of farmers is modelled by a nonlinear porous medium type diffusion equation which coincides with the singular limit of another model for the dispersal of farmers as a small parameter tends to zero. From the ecological viewpoint, the nonlinear diffusion takes into account the population density pressure of the farmers on their dispersal. The interaction between farmers and hunter-gatherers is of the Lotka-Volterra prey-predator type. We show the existence and uniqueness of a global in time solution and study its asymptotic behaviour as time tends to infinity.

**AMS subject classifications**: 35K57, 35Q92, 92D40

**Key words**: Farmer–hunters model, reaction–diffusion system, degenerate diffusion, existence and uniqueness of the solution, exponential convergence to equilibrium.

## 1 Introduction

The Neolithic migration of farmers in regions previously inhabited by hunter-gatherers has been studied for a long time [1,2]. In particular the Lotka-Volterra type system

$$\begin{cases} F_t = d_F \Delta F + r_F F(1-F+aH), \\ H_t = d_H \Delta H + r_H H(1-H-bF), \end{cases} \tag{1.1}$$

*Corresponding author. *Email addresses:* `jan.elias@uni-graz.at` (J. Eliaš), `danielle.hilhorst@math.u-psud.fr` (D. Hilhorst), `mimura.masayasu@gmail.com` (M. Mimura)

has been studied by [3]. In this model, the populations of farmers $F$ and hunter-gatherers $H$ are assumed to diffuse freely and randomly by linear diffusion with constant diffusion rates $d_F$ and $d_H$ throughout the region. Recently, Eliaš, *et al.* [15] have proposed a new three component reaction-diffusion system

$$\begin{cases} F_{1,t} = F_1(1-F_1-F_2)+sF_1H-k(p(F_1+F_2)F_1-(1-p(F_1+F_2))F_2), \\ F_{2,t} = d\Delta F_2+F_2(1-F_1-F_2)+sF_2H+k(p(F_1+F_2)F_1-(1-p(F_1+F_2))F_2), \\ H_t = \Delta H+rH(1-H)-g(F_1+F_2)H, \end{cases} \tag{$\mathcal{P}^k$}$$

allowing to monitor the expanding farming population in terms of the sedentary and migrating farmers denoted by $F_1$ and $F_2$, respectively. In Problem $(\mathcal{P}^k)$, $p=p(F)$ is the probability density function which is included in the switching mechanism between the sedentary and migrating farmers and which depends on the total density of the farmers $F=F_1+F_2$. We assume that $p$ satisfies

$$\begin{cases} (i) & p(0)=0, \\ (ii) & p(F) \text{ is increasing in } F, \\ (iii) & \lim\limits_{F\to\infty} p(F)=1. \end{cases}$$

A simple example is given by $p(F)=F/(F+F_c)$, where $F_c$ is the switching value of the conversion between $F_1$ and $F_2$; more precisely, the probabilities of remaining sedentary or migrating are both equal to $1/2$ when $F=F_c$. Finally the parameter $k>0$ is the rate of conversion between $F_1$ and $F_2$. In view of (i)-(iii), the model $(\mathcal{P}^k)$ implies that whenever the total density of farmers is low, the farmers prefer a sedentary lifestyle. On the other hand, if the total density of farmers is high, then some of the farmers start migrating and searching for new places favourable for sedentary life.

In this paper, we consider the special case when the rate of conversion $k$ in Problem $(\mathcal{P}^k)$ tends to $\infty$. Formal calculations show and we will prove in a forthcoming article [14] that $(F_{k,1}+F_{k,2},H_k)$ converges to $(F,H)$ as $k\to\infty$, where the triple $(F_{k,1},F_{k,2},H_k)$ satisfies Problem $(\mathcal{P}^k)$ and $(F,H)$ is a solution of the system

$$\begin{cases} F_t = d_F\Delta(p(F)F)+r_FF(1-F)+sFH, \\ H_t = d_H\Delta H+r_HH(1-H)-gFH. \end{cases} \tag{1.2}$$

Unlike in the system (1.1), the diffusion of farmers may degenerate if $p=0$ in (cf. assumption (i)). In this model, the Neolithic dispersal of farming in Europe takes into account the population density pressure due to limited space and the advanced lifestyle resulting in farmer overcrowding.

More precisely, we study the nondimensionalised model

$$\begin{cases} u_t = d_u\Delta\varphi(u)+r_uu(1-u+av) & \text{in } Q_T, \\ v_t = d_v\Delta v+r_vv(1-v-bu) & \text{in } Q_T, \\ \frac{\partial\varphi(u)}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0 & \text{on } \Sigma_T, \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x) & x\in\Omega, \end{cases} \tag{$\mathcal{P}$}$$