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Decay and Scattering of Solutions to Nonlinear Schrödinger Equations with Regular Potentials for Nonlinearities of Sharp Growth

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Abstract. In this paper, we prove the decay and scattering in the energy space for nonlinear Schrödinger equations with regular potentials in $\mathbb{R}^d$ namely, $i\partial_t u+\Delta u-V(x)u+\lambda|u|^{p-1}u=0$. We will prove decay estimate and scattering of the solution in the small data case when $1+\frac{2}{d}<p\le 1+\frac{4}{d-2}$, $d\ge 3$. The index $1+\frac{2}{d}$ is sharp for scattering concerning the result of Strauss [22]. This result generalizes the one-dimensional work of Cuccagna *et al.* [4] to all $d\ge 3$.

AMS subject classifications: 35Q55

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1 Introduction

In this paper, we consider the nonlinear Schrödinger equation with a potential:

$$\begin{cases} i\partial_t u+\Delta_V u+\lambda|u|^{p-1}u=0, \\ u(h,x)=u_0(x), \end{cases} \tag{1.1}$$

where $u:[h,\infty)\times\mathbb{R}^d\to\mathbb{C},h>0$, $\Delta_V=\Delta-V,V:\mathbb{R}^d\to\mathbb{R}$, $\lambda=\pm 1$ and $1<p<\infty$. When $p=1+\frac{4}{d},d\ge 1$ and $1+\frac{4}{d-2},d\ge 3$, the equation is called mass-critical and energy-critical respectively. The equation is called mass-supercritical for $d\ge 1$ if $p>1+\frac{4}{d}$ and energy-subcritical for $d\ge 3$ if $p<1+\frac{4}{d-2}$. If $\lambda=-1$, the equation is called defocusing; otherwise, it is called focusing if $\lambda=1$.

There are many important areas of application which motivate the study of nonlinear Schrödinger equations with potentials (Gross-Pitaevskii equation). In the most fundamental level, it arises as a mean field limit model governing the interaction of a plenty

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large number of weakly interacting bosons [13,16,21]. In a macroscopic level, it arises as the equation governing the evolution of the envelope of the electric field of a light pulse propagating in a medium with defects, see for instance, [10,11].

In this paper, we aim to prove scattering results for (1.1) as what has been done in the case of nonlinear Schrödinger equations without potentials. More precisely, we will show the scattering in the small data case for any $1+\frac{2}{d}<p\leq 1+\frac{4}{d-2}$, $d\geq 3$, which is sharp concerning the result of Strauss [22].

The decay and scattering for small initial data have been studied for decades. When $V=0$, it has been shown that for $d\geq 1$, $p=1+\frac{2}{d}$ is the critical exponent for scattering. In fact, for $1+\frac{2}{d}<p<1+\frac{4}{d}$, $d\geq 1$, decay and scattering of the solution in the small data case were proved by McKean and Shatah [17]. For $1+\frac{4}{d}\leq p\leq 1+\frac{4}{d-2}$, $d\geq 3$ and $1+\frac{4}{d}\leq p<\infty$, $d=1,2$, local wellposedness and small data scattering were proved by Strauss [23], see Visciglia [24] for further development and [3] for more reference. Moreover, Strauss [22] showed when $1<p\leq 1+\frac{2}{d}$ for $d\geq 2$ and $1<p\leq 2$ for $d=1$, the only scattering solution is zero. This was extended to the case $1<p\leq 3$ for $d=1$ by Barab [1]. The existence and the form of the scattering operator were obtained by Ozawa [18] for $d=1$ and by Ginibre and Ozawa [7] for $d\geq 2$. The completeness of the scattering operator and the decay estimate were obtained by Hayashi and Naumkin [12]. For all solutions, not only for small ones, for $d=1$ in defocusing case, the completeness of the scattering operator and decay were obtained by Deift and Zhou [6].

When $V\neq 0$, the situation is much more involved. In [4], Cuccagna, Georgiev and Visciglia proved decay and scattering for small initial data for $p>3$ in one dimension when the potential V is a real Schwartz function with assumptions on its scattering matrix.

In the article, we will consider potentials satisfying the following assumptions:

Regular Potential Hypothesis

Suppose that V is a real-valued potential satisfying

(i) $\langle x\rangle^{2N+1}(|V(x)|+|\nabla V(x)|)\in L^\infty(\mathbb{R}^d)$, for some $N>d$;

(ii) the spectrum of $-\Delta_V$ is continuous, and 0 is neither a resonance nor an eigenvalue of $-\Delta_V$;

(iii) $\langle x\rangle^\alpha V(x)$ is a bounded operator from H^η to H^η for some $\alpha>d+4$, $\eta>0$ with $\mathcal{F}V\in L^1$;

Define $\|f\|_\Sigma=\|\langle x\rangle^2 f\|_2+\|\langle x\rangle\nabla f\|_2+\|f\|_{H^2}$. Our main theorem is as follows which extends the results of [4] to higher dimensions ($d\geq 3$).

Theorem 1.1. *Suppose that V satisfies the Regular Potential Hypothesis and its Kato norm satisfies*

$$\sup_x\int\frac{|V(y)|}{|x-y|^{d-2}}\mathrm{d}y<\frac{1}{c(d)}, \tag{1.2}$$

where $c(d)$ is a constant in the estimate of the Bessel function of the third type and $c(3)=2\pi$. For $d\geq 3$, $1+\frac{2}{d}<p\leq 1+\frac{4}{d-2}$, $\lambda=\pm 1$, if the initial data $\|u_0\|_\Sigma$ is sufficiently small, then (1.1) *is*