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A Well-Balanced Discontinuous Galerkin Method for the Blood Flow Model

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Abstract. Numerical simulations by high order methods for the blood flow model in arteries have wide applications in medical engineering. This blood flow model admits the steady state solutions, for which the flux gradient is non-zero, and is exactly balanced by the source term. In this paper, we design a high order discontinuous Galerkin method to this model by means of a novel source term approximation as well as well-balanced numerical fluxes. Rigorous theoretical analysis as well as extensive numerical results all suggest that the resulting method maintains the well-balanced property, enjoys high order accuracy and keeps good resolutions for smooth and discontinuous solutions.

AMS subject classifications: 65M60

Key words: Blood flow model, discontinuous Galerkin method, well-balanced property, high order accuracy, source term.

1 Introduction

Numerical simulations by high order methods for the blood flow model in arteries have wide applications in medical engineering [1,2] because of the pulsatility of blood and the necessity to catch properly the waves propagation in arteries. In one space dimension, the blood flow model takes the form of the following system of hyperbolic balance laws [3]:

$$\begin{cases} A_t+Q_x=0, \\ Q_t+\left(\frac{Q^2}{A}+\frac{K}{3\rho\sqrt{\pi}}A^{\frac{3}{2}}\right)_x=\frac{KA}{2\rho\sqrt{\pi}\sqrt{A_0}}(A_0)_x, \end{cases} \tag{1.1}$$

where $A=\pi R^2$ (R being the radius of the vessel) is the cross-sectional area, $Q=Au$ denotes the discharge, u means the flow velocity, ρ stands for the blood density and K represents the arterial stiffness. In addition, $A_0=\pi R_0^2$ is the cross section at rest ($u=0m/s$) with

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R_0 being the radius of the vessel, which may be variable in the case of aneurism, stenosis or taper. This model for blood flow can be derived with some simplifying hypothesis from the Navier-Stokes equations [4,5].

There exists a steady state for the above system (1.1), also called mechanical equilibrium:

$$u=0 \text{ and } A=A_0. \tag{1.2}$$

Under the above steady state (1.2), the flux gradient is non-zero and is exactly balanced by the source term. Consequently, it is desirable to maintain the balance between the flux gradient and the source term at the discrete level. In general, standard numerical methods may not satisfy the discrete version of this balance exactly at (or near) the steady state, and even introduce spurious oscillations, unless the mesh is extremely refined. But the mesh refinement procedure is not applicable for the high-dimensional problems due to very high computational cost. In order to save the computational cost, well-balanced methods are specially introduced to preserve exactly these steady state solutions up to machine accuracy [6]. In addition, well-balanced methods [6] can capture small perturbations on relatively coarse meshes [7]. More information about well-balanced methods can be found in the lecture note [8].

From the numerical point of view, there are many attempts based on the numerical methods for the blood flow model in the literature, e.g., [9,10]. In recent years, there have been many interesting attempts on the well-balanced methods. For example, Delestre and Lagrée [11] presented a well-balanced finite volume scheme for the blood flow model based on the conservative governing equations [12–14]. Müller *et al.* [15] constructed a high order well-balanced finite volume scheme for the blood flow in elastic vessels with various mechanical properties. Recently, Murillo *et al.* [16] presented an energy-balanced approximate solver for the blood flow model with upwind discretization for the source term. More recently, Wang *et al.* [17] designed a well-balanced finite difference weighted essentially non-oscillatory (WENO) scheme.

All of the works mentioned above are under the framework of finite difference schemes or finite volume schemes. During the past few decades, high order discontinuous Galerkin (DG) methods have gained great attention in solving hyperbolic conservation laws. DG method is a class of finite element methods using discontinuous piecewise polynomial space as the solution and test function spaces (see [18] for a historic review). It combines advantages of both finite element methods and finite volume methods, and can achieve high order accuracy easily with the use of high order polynomials within each element. Several advantages of the DG methods, including its accuracy, high parallel efficiency, flexibility for hp-adaptivity and arbitrary geometry and meshes, make it useful for a wide range of applications.

The key objective of this paper is to develop a high order accurate well-balanced DG method for the blood flow model with the help of a novel source term approximation as well as well-balanced numerical fluxes. The resulting DG method preserves the well-balanced property and keeps the original high order of accuracy. To our knowledge,