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# New Alternately Linearized Implicit Iteration for M-matrix Algebraic Riccati Equations

Jinrui Guan[1,*] and Linzhang Lu[2]

[1] *Department of Mathematics, Taiyuan Normal University, Shanxi 030619, P.R. China.*
[2] *School of Mathematics and Computer Science, Guizhou Normal University, Guizhou 550001, P.R. China.*

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**Abstract.** Research on the theories and the efficient numerical methods of M-matrix algebraic Riccati equation (MARE) has become a hot topic in recent years due to its broad applications. In this paper, based on the alternately linearized implicit iteration method (ALI) [Z.-Z. Bai *et al.*, Numer. Linear Algebra Appl., 13(2006), 655-674.], we propose a new alternately linearized implicit iteration method (NALI) for computing the minimal nonnegative solution of M-matrix algebraic Riccati equation. Convergence of the NALI method is proved by choosing proper parameters for the MARE associated with nonsingular M-matrix or irreducible singular M-matrix. Theoretical analysis and numerical experiments show that the NALI method is more efficient than the ALI method in some cases.

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**Key words**: M-matrix algebraic Riccati equation, M-matrix, ALI iteration method.

## 1 Introduction

The nonsymmetric algebraic Riccati equation (NARE) is of the form

$$XCX-XD-AX+B=0, \tag{1.1}$$

where $A$, $B$, $C$ and $D$ are real matrices of sizes $m\times m$, $m\times n$, $n\times m$ and $n\times n$ respectively. For (1.1), let

$$K=\begin{pmatrix} D & -C \\ -B & A \end{pmatrix}. \tag{1.2}$$

If $K$ is an M-matrix, then (1.1) is called an M-matrix algebraic Riccati equation (MARE). M-matrix algebraic Riccati equation arises from many branches of applied mathematics,

*Corresponding author. *Email addresses:* `guanjinrui2012@163.com` (J. Guan), `llz@gznu.edu.cn` (L. Lu)

 

such as transport theory, Wiener-Hopf factorization of Markov chains, stochastic process, and so on [2,3,5,7,14,18,20]. Research on the theories and the efficient numerical methods of MARE has become a hot topic in recent years. The solution of practical interest is the minimal nonnegative solution. For theoretical background we refer to [5,7,8,10–12,15].

The following are some notations and definitions we need in the sequel.

For any matrices $A=(a_{ij}),B=(b_{ij})\in\mathbb{R}^{m\times n}$, we write $A\geq B(A>B)$, if $a_{ij}\geq b_{ij}(a_{ij}>b_{ij})$ for all $i,j$. $A$ is called a Z-matrix if $a_{ij}\leq 0$ for all $i\neq j$. A Z-matrix $A$ is called an M-matrix if there exists a nonnegative matrix $B$ such that $A=sI-B$ and $s\geq\rho(B)$ where $\rho(B)$ is the spectral radius of $B$. In particular, $A$ is called a nonsingular M-matrix if $s>\rho(B)$ and singular M-matrix if $s=\rho(B)$.

We review some basic results of M-matrix. The following lemmas can be found in [4, Chapter 6].

**Lemma 1.1.** *Let $A$ be a Z-matrix. Then the following statements are equivalent:*
*(1) $A$ is a nonsingular M-matrix;*
*(2) $A^{-1}\geq 0$;*
*(3) $Av>0$ for some vectors $v>0$;*
*(4) All eigenvalues of $A$ have positive real part.*

**Lemma 1.2.** *Let $A$ and $B$ be Z-matrices. If $A$ is a nonsingular M-matrix and $A\leq B$, then $B$ is also a nonsingular M-matrix. In particular, for any nonnegative real number $\alpha$, $B=\alpha I+A$ is a nonsingular M-matrix.*

**Lemma 1.3.** *Let $A$ be an M-matrix, $B\geq A$ be a Z-matrix. If $A$ is nonsingular or irreducible singular and if $A\neq B$, then $B$ is also a nonsingular M-matrix.*

**Lemma 1.4.** *Let $A$ be a nonsingular M-matrix or an irreducible singular M-matrix. Let $A$ be partitioned as*

$$A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

*where $A_{11}$ and $A_{22}$ are square matrices. Then $A_{11}$ and $A_{22}$ are nonsingular M-matrices.*

**Lemma 1.5.** *If $A$, $B$ are nonsingular M-matrices and $A\leq B$, then $A^{-1}\geq B^{-1}$.*

For the minimal nonnegative solution of the MARE, we have the following important result [5,7,8,12].

**Lemma 1.6.** *If $K$ is a nonsingular M-matrix or an irreducible singular M-matrix, then (1.1) has a unique minimal nonnegative solution $S$. If $K$ is nonsingular, then $A-SC$ and $D-CS$ are also nonsingular M-matrices. If $K$ is irreducible, then $S>0$ and $A-SC$ and $D-CS$ are also irreducible M-matrices.*

There are many numerical methods up to now proposed for the minimal nonnegative solution of MARE, such as Schur method, matrix sign function, fixed-point iteration, Newton iteration, doubling algorithms, and so on. For details see [1,5–7,9,13,16,17,19].