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Hermite Expansion of the Riemann Zeta Function

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Abstract. Let $\zeta(s)$ be the Riemann zeta function, $s=\sigma+it$. For $0<\sigma<1$, we expand $\zeta(s)$ as the following series convergent in the space of slowly increasing distributions with variable t :

$$\zeta(\sigma+it)=\sum_{n=0}^{\infty}a_n(\sigma)\psi_n(t),$$

where

$$\psi_n(t)=(2^n n!\sqrt{\pi})^{-1/2}e^{\frac{-t^2}{2}}H_n(t),$$

$H_n(t)$ is the Hermite polynomial, and

$$a_n(\sigma)=2\pi(-1)^{n+1}\psi_n(i(1-\sigma))+(-i)^n\sqrt{2\pi}\sum_{m=1}^{\infty}\frac{1}{m^\sigma}\psi_n(\ln m).$$

This paper is concerned with the convergence of the above series for $\sigma>0$. In the deduction, it is crucial to regard the zeta function as Fourier transfomations of Schwartz' distributions.

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1 Results

Let $\zeta(s)$ be the famous Riemann Zeta function which is holomorphic for $s=\sigma+it\in\mathcal{C}-\{1\}$. It is well known that if $0<\sigma<1$ then

$$\zeta(s)=s\int_0^\infty\frac{[x]-x}{x^{s+1}}dx$$

(see [1] or [2]). By the substitute of variables $x=e^y$, we get

$$\zeta(s)=s\int_{-\infty}^{\infty}([e^y]-e^y)e^{-sy}dy.$$

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Set

$$f(y)=[e^y]-e^y. \tag{1.1}$$

Then for $0<\sigma<1$, $e^{-\sigma y}f(y)$ is a slowly increasing function , so can be regarded as an element of $\mathcal{S}'$, the dual space of the space $\mathcal{S}$ of rapidly decreasing functions on R. The Laplace transformation $\mathcal{L}(f)(s)$ of f is then defined on the trip $0<\sigma<1$ both in the ordinary and distributional sense, that is

$$\mathcal{L}(f)(s)=\zeta(s)/s.$$

Let $f'\in\mathcal{S}'$ be the derivative of f in the distributional sense. then $e^{-\sigma y}f'(y)\in\mathcal{S}'$ for $0<\sigma<1$, and Laplace transformation $\mathcal{L}(f')(s)$ is defined on the strip $0<\sigma<1$ such that

$$\mathcal{L}(f')(s)=\zeta(s)$$

([3] Chapter 8). So by the relation of Fourier and Laplace transformation of distribution (see also [3]), we see that $\zeta(\sigma+it)$ as function of t is the Fourier transformation of $e^{-\sigma y}f'(y)$ in the distributional sense, where the Fourier transformation is defined by $g(x)\to\int_{-\infty}^{\infty}g(x)e^{-ixy}dx$ for $g\in L^1(R)$.

Recall that the Hermite polynomials are defined as

$$H_n(x)=e^{x^2}(\frac{d}{dx})^n e^{-x^2},\ n=0,1,\dots$$
$$\psi_n(t)=(2^n n!\sqrt{\pi})^{-1/2}e^{\frac{-t^2}{2}}H_n(t),\ n=0,1,\dots$$

form a complete normalized orthogonal system in $L^2(R)$.

Xiaqi Ding and his collaborators introduced and developed the theory of Hermite expansions of generalized functions [4]. The aim of this paper is to give the Hermite expansion of $\zeta(\sigma+it)$ as function of t for $0<\sigma<1$. For this, we give first the Hermite expansion of $e^{-\sigma y}f'(y)\in\mathcal{S}'$ for fixed σ. Now

$$f'(y)=-e^y+\sum_{m=1}^{\infty}\delta(y-\ln m),$$

where δ is the Dirac δ-function. So

$$e^{-\sigma y}f'(y)=-e^{(1-\sigma)y}+\sum_{m=1}^{\infty}\frac{1}{m^{\sigma}}\delta(y-\ln m). \tag{1.2}$$

The following lemma gives the Hermite expansion of $-e^{(1-\sigma)y}$.

Lemma 1.1. *For any complex number a,*

$$\int_{-\infty}^{\infty}e^{ax}\psi_n(x)dx=(-i)^n\sqrt{2\pi}\psi_n(ia).$$