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A Nonhomogeneous Boundary Value Problem for the Boussinesq Equation on a Bounded Domain

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Abstract. In this paper, we study the well-posedness of an initial-boundary-value problem (IBVP) for the Boussinesq equation on a bounded domain,

$$\begin{cases} u_{tt}-u_{xx}+(u^2)_{xx}+u_{xxxx}=0, \quad x\in(0,1),\, t>0,\\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x),\\ u(0,t)=h_1(t),\ u(1,t)=h_2(t),\ u_{xx}(0,t)=h_3(t),\ u_{xx}(1,t)=h_4(t). \end{cases}$$

It is shown that the IBVP is locally well-posed in the space $H^s(0,1)$ for any $s\geq 0$ with the initial data φ, ψ lie in $H^s(0,1)$ and $H^{s-2}(0,1)$, respectively, and the naturally compatible boundary data h_1, h_2 in the space $H_{loc}^{(s+1)/2}(\mathbb{R}^+)$, and h_3, h_4 in the the space of $H_{loc}^{(s-1)/2}(\mathbb{R}^+)$ with optimal regularity.

AMS subject classifications: 35Q53, 35B65

Key words: Boussinesq equation, initial-boundary value problem, local well-posedness.

1 Introduction

In this article, we consider a nonhomogeneous boundary value problem for the Boussinesq equation posed on a bounded domain $(0,1)$,

$$\begin{cases} u_{tt}-u_{xx}+(u^2)_{xx}+u_{xxxx}=0, \quad x\in(0,1),\, t>0,\\ u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x),\\ u(0,t)=h_1(t),\ u(1,t)=h_2(t),\ u_{xx}(0,t)=h_3'(t),\ u_{xx}(1,t)=h_4'(t). \end{cases} \tag{1.1}$$

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Here $h'(t)$ denotes the derivative of $h(t)$. Equation of this type, but with negative sign for the fourth derivative term,

$$u_{tt}-u_{xx}+(u^2)_{xx}-u_{xxxx}=0, \tag{1.2}$$

was originally derived in 1871 by J. Boussinesq in his study [9] on propagation of small amplitude, long waves on the surface of water. The Boussinesq equation (1.2) possesses special traveling wave solutions called the solitary wave, and it is the first equation that gives a mathematical explanation to the phenomenon of solitary waves discovered by Scott Russell reported in 1834. The original Boussinesq equation has been used in a considerable range of applications such as coasts and harbors engineering, simulation of tides and tsunamis.

However, equation (1.2) is ill-posed for its initial-value problem. That is, a slight difference in initial data, might evolve to a large change in solution. For this reason, the equation (1.2) is sometimes called as the "bad" Boussinesq equation. This can be seen, for example, by considering its linear equation as

$$u_{tt}-u_{xxxx}:=(\partial_t+\partial_{xx})(\partial_t-\partial_{xx})u=0.$$

The "$\partial_t-\partial_{xx}$" can be treated as the heat equation and it is well-posed, but "$\partial_t+\partial_{xx}$", the backward heat equation, is ill-posed. One way to correct this ill-posedness issue is to alter the sign of the fourth derivative term, then the "good" Boussinesq equation

$$u_{tt}-u_{xx}+(u^2)_{xx}+u_{xxxx}=0, \tag{1.3}$$

is proposed. Similarly, its well-posedness can be seen by considering the linear equation

$$u_{tt}+u_{xxxx}:=(\partial_t+i\partial_{xx})(\partial_t-i\partial_{xx})u=0,$$

because both the Schrödinger and the reversed Schrödinger equation are indeed well-posed.

As the changing of sign to the "bad" Boussinesq equation, the "good" Boussinesq equation cannot be justified as the original Boussinesq's physical modeling. However, Zakharov [36] has proposed it as a model of nonlinear vibration along a string, and Turitsyn [27] has revealed it when describing electromagnetic waves in nonlinear dielectric materials. Moreover, the "good" Boussinesq equation appeals in the study of Falk et al [11] as a model of shape-memory alloys and it is also raised in a large range of physical phenomena including propagation of ion-sound waves in a plasma and nonlinear lattice waves.

The study of the well-posedness of the initial-value problem (IVP) of the Boussinesq equation,

$$\begin{cases} u_{tt}-u_{xx}+(u^2)_{xx}+u_{xxxx}=0, & x\in\mathbb{R},\ t\in\mathbb{R}, \\ u(x,0)=f(x), \qquad u_t(x,0)=h(x), \end{cases}$$