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Singular Solutions of a Boussinesq System for Water Waves

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Abstract. Studied here is the Boussinesq system

$$\begin{aligned}&\eta_t+u_x+(\eta u)_x+au_{xxx}-b\eta_{xxt}=0,\\&u_t+\eta_x+\frac{1}{2}(u^2)_x+c\eta_{xxx}-du_{xxt}=0,\end{aligned}$$

of partial differential equations. This system has been used in theory and practice as a model for small-amplitude, long-crested water waves. The issue addressed is whether or not the initial-value problem for this system of equations is globally well posed. The investigation proceeds by way of numerical simulations using a computer code based on a a semi-implicit, pseudo-spectral code. It turns out that larger amplitudes or velocities do seem to lead to singularity formation in finite time, indicating that the problem is not globally well posed.

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1 Introduction

A class of multi-dimensional Boussinesq systems of the form

$$\begin{aligned}&\eta_t+\nabla\cdot\boldsymbol{v}+\nabla\cdot\eta\boldsymbol{v}+a\Delta\nabla\cdot\boldsymbol{v}-b\Delta\eta_t=0,\\&\boldsymbol{v}_t+\nabla\eta+\frac{1}{2}\nabla|\boldsymbol{v}|^2+c\Delta\nabla\eta-d\Delta\boldsymbol{v}_t=0,\end{aligned}\tag{1.1}$$

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was put forward in [6] as models for the propagation of small amplitude, long wavelength surface water waves. Here, $\eta=\eta(x,y,t)$ describes the height of the free surface, relative to its rest position, above the point $(x,y,0)$ on the bottom and the vector $\boldsymbol{v}=\boldsymbol{v}(x,y,\theta,t)$ has components (u,w) representing the horizontal velocity field at height θ above the same point. The gradient, divergence and Laplacian are all taken with respect to the horizontal spatial variables (x,y). The overlying assumptions leading to such models are that the ratio α of the waveheight to the undisturbed depth be small, the ratio β of the undisturbed depth to a typical wavelength be small, but that the quotient α/β^2 be of order one. As written in (1.1), the horizontal coordinates x and y are measured in wavelengths whereas the vertical coordinate z is measured in depths. A consequence of the latter fact is that $\theta\in[0,1]$. The constants a,b,c,d have the form

$$a=\left(\frac{\theta^2}{2}-\frac{1}{6}\right)\lambda,\quad b=\left(\frac{\theta^2}{2}-\frac{1}{6}\right)(1-\lambda),\quad c=\frac{1-\theta^2}{2}\nu,\quad d=\frac{1-\theta^2}{2}(1-\nu), \tag{1.2}$$

where λ and ν are modeling parameters which may take any real value without disturbing the formal level of approximation inherent in such models. Details of the derivation and scaling being used may be found in [6,7,12].

The system (1.1) of three, coupled, nonlinear evolution equations allows for propagation in all directions. The long-crested regime is where most of the motion takes place in the x–direction, say, with little or no variation in the y–direction. In this case, the model simplifies. The second component w of the horizontal velocity $\boldsymbol{v}$ is zero, derivatives with respect to y vanish and the third equation is satisfied identically. The system then reduces to

$$\begin{aligned}&\eta_t+u_x+(\eta u)_x+au_{xxx}-b\eta_{xxt}=0,\\&u_t+\eta_x+\frac{1}{2}(u^2)_x+c\eta_{xxx}-du_{xxt}=0,\end{aligned} \tag{1.3}$$

where a,b,c and d are as above and subscripts connote partial differentiation.

Mathematical theory for various of both the one– and two–dimensional versions of this class of Boussinesq systems appears in [1,3,5–7,12,21–25], for example. In particular, combining the results in [12], the existence theory for the model equations mentioned above and similar results for the full water-wave problem in [2], one sees that these systems are indeed approximations of the full, inviscid, water wave problem with rigorous error estimates that validate the formal asymptotics that go into their derivation.

The results in the papers about the Boussinesq models reveal that some of the *abcd*-systems are well posed, at least locally in time. These are the candidates for use in practical situations. However, for almost all of these systems, there is little information available as to whether or not they are globally well posed, even for small data. (The exceptions are what is termed the original Boussinesq system, see [25] and [3], which is globally well posed for arbitrary-sized, localized, smooth initial data and the Bona-Smith system [10] which is globally well posed for order-one initial data.