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Elliptic Systems with a Partially Sublinear Local Term

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Abstract. Let $1<p<2$. Under some assumptions on V,K, existence of infinitely many solutions $(u,\phi)\in H^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$ is proved for the Schrödinger-Poisson system

$$\begin{cases} -\Delta u+V(x)u+\phi u=K(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi=u^2 & \text{in } \mathbb{R}^3 \end{cases}$$

as well as for the Klein-Gordon-Maxwell system

$$\begin{cases} -\Delta u+[V(x)-(\omega+e\phi)^2]u=K(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi+e^2u^2\phi=-e\omega u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where ω, $e>0$. This is in sharp contrast to D'Aprile and Mugnai's non-existence results.

AMS subject classifications: 35A15, 35J50

Key words: Schrödinger-Poisson system, Klein-Gordon-Maxwell system, infinitely many solutions.

1 Introduction and main results

In this paper, we study existence of infinitely many solutions $(u,\phi)\in H^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$ to the Schrödinger-Poisson system

$$\begin{cases} -\Delta u+V(x)u+\phi u=K(x)|u|^{p-2}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi=u^2 & \text{in } \mathbb{R}^3 \end{cases} \tag{1.1}$$

for $1<p<2$.

This system has a wide background in physics. It is reduced from the Hartree-Fock equations by a mean field approximation ([9, 10]). It also describes the Klein-Gordon or

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Schrödinger fields interacting with an electromagnetic field ([3]). The related Thomas-Fermi-von Weizsäcker model describes the ground states of nonrelativistic atoms and molecules in the quantum mechanics ([1]).

In [2], D'Aprile and Mugnai prove that if $V\equiv 1\equiv K$ then (1.1) has no nontrivial solution. In the present paper we prove that if V is a potential well and K is positive somewhere in $\mathbb{R}^3$ then (1.1) has infinitely many nontrivial solutions. To be more precise, as a special case of our main results, we will show that the system has infinitely many solutions provided that $V,K\in C(\mathbb{R}^3,\mathbb{R})$, $\inf V>-\infty$, there is $R>0$ such that $V(x)>0$ for $|x|\geq R$, $\int_{|x|\geq R}V^{-1}<\infty$, K is bounded, and there exists $x_0\in\mathbb{R}^3$ such that $K(x_0)>0$. In fact, one of our main theorems states a much more general result for a more general system.

We will consider the more general system

$$\begin{cases} -\Delta u+V(x)u+\phi u=f(x,u) \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi=u^2 \quad \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

To state our main result, we need the following assumptions:

(V) $V\in C(\mathbb{R}^3,\mathbb{R})$, $\inf V>-\infty$, there is $R>0$ such that

$$V(x)>0 \ \ for \ \ |x|\geq R, \quad \int_{|x|\geq R}V^{-1}<\infty.$$

(F) There exist positive numbers δ and c and $p\in(1,2)$ such that $f\in C(\mathbb{R}^3\times[-\delta,\delta],\mathbb{R})$, $f(x,t)$ is odd in t,

$$|f(x,t)|\leq c|t|^{p-1} \ \ for \ \ |t|\leq\delta,$$

and there exist $x_0\in\mathbb{R}^3$ and $r>0$ such that

$$\lim_{t\to 0}F(x,t)/t^2=\infty$$

uniformly in $x\in B_r(x_0):=\{x\in\mathbb{R}^3 \mid |x-x_0|<r\}$, where $F(x,t)=\int_0^t f(x,s)ds$.

Theorem 1.1. *Under* (V) *and* (F), (1.2) *has infinitely many nontrivial solutions in* $H^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$.

Assumption (V) makes V look like a well-shaped potential. Note that the nonlinear term $f(x,t)$ in assumption (F) is defined only for $|t|\leq\delta$. Accordingly, the $L^\infty(\mathbb{R}^3)$ norm of u in (u,ϕ), the solution we will obtain, will have to be less than δ.

From (V) and (F), it is without loss of any generality to assume further in Theorem 1.1 that

$$\inf V>0 \quad \text{and} \quad \int_{\mathbb{R}^3}V^{-1}<\infty. \tag{1.3}$$