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An Adaptive Grid Method for a Non-Equilibrium PDE Model from Porous Media

Paul Andries Zegeling*

Department of Mathematics, Faculty of Science, Utrecht University, the Netherlands

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Abstract. An adaptive grid method is applied to a PDE model from geo-hydrology. Due to the higher mixed-order derivative, non-monotone waves can appear which could represent similar structures as observed in laboratory experiments [5, 16, 18]. The effectiveness of the adaptive grid, which is based on a smoothed equidistribution principle, is shown compared to uniform grid simulations. On a uniform grid (numerical) oscillating non-monotone waves may appear which are not present in the adaptive grid.

AMS subject classifications: 35Mxx, 35M11, 65Mxx, 65M20, 65M50

Key words: Adaptive grids, Method of lines, Dynamical systems, Travelling waves, Monotonicity, Porous media, Water saturation.

1 Introduction

Non-monotone waves play an important role in geo-hydrology. In particular, these phenomena are observed in laboratory experiments with, so-called, saturation overshoot and fingering structures. Figure 1 shows such behaviour in two-phase porous media, similar to the solutions and structures observed in [5,16,18]. The left frame illustrates saturation profiles for different values of the injection rate of the water at the left boundary of the domain, starting with initially dry sand in the tube. This corresponds with an almost one-dimensional domain in terms of the underlying PDE model. We can see monotone waves (for low injection rates), non-monotone waves (for average injection rates) and plateau-type waves (for high injection rates). The second frame depicts fingering behaviour in a higher-dimensional setting, similar to the solutions described in [16], in which the black spots indicate the overshoots compared to the blue regions where monotone 'flat' waves exist. It is known from PDE theory that the traditional models to describe flows in porous media, such as Richards' equation or the Buckley-Leverett equation, only possess monotone waves [9,10,12]. To deal with this problem, Hassanizadeh and Gray [11] proposed

*Corresponding author. *Email addresses:* `P.A.Zegeling@uu.nl` (P.-A. Zegeling)

a PDE model, using a thermodynamics approach, with an additional term in the model to include non-static conditions for the capillary pressure. Such a model which includes a mixed higher-order term, gives rise to travelling waves (TWs) with non-monotone profiles. This behaviour can be explained using an analysis from dynamical systems theory in which stationary points in the phase plane are connected by a special curve, called the separatrix. To support the theoretical considerations, numerical experiments for the PDE model are performed. For an accurate treatment of the numerical PDE solution, it is of importance to approximate the steep waves by an adaptive grid method. To my knowledge, no adaptive methods have been applied yet to this non-equilibrium PDE model. Only a few articles in literature deal with adaptive methods applied to porous media (either for the traditional theory with monotone waves or for other theoretical models with non-monotone waves): [7,8,13]. For an efficient application of adaptive grid methods, it is known from literature ([1,3,4,14,15,19–21]) that smoothness of the non-uniform grid in the space- <u>and</u> time-direction is crucial. Therefore, the adaptive grid method in this paper is enhanced with smoothing operators both in space and time.

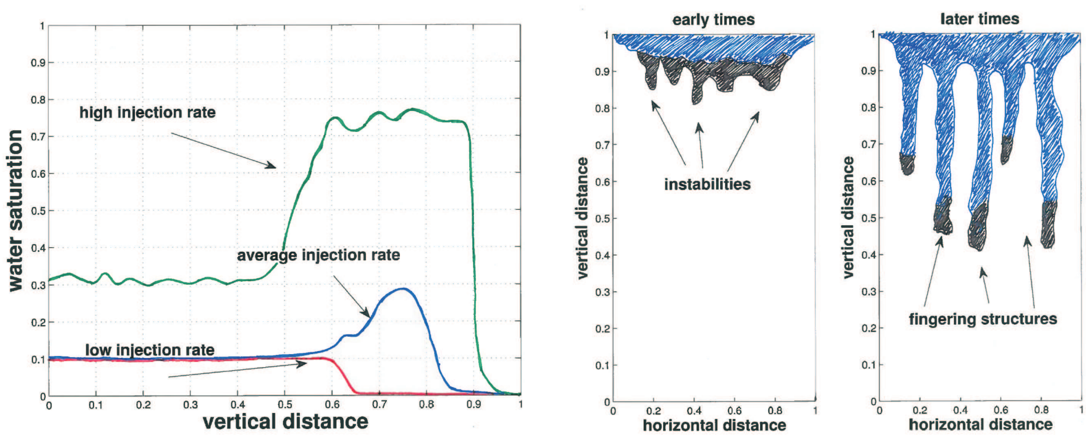

Figure 1: A sketch of two different types of laboratory experiments with non-monotone waves and fingering behaviour (left frame: monotone, non-monotone and plateau waves, similar to the ones observed in [5,18] and right frame: the occurence of instabilities and fingering behaviour, similar to the observations in [16]).

The paper is organized as follows. Section 2 introduces the non-equilibrium PDE, the so-called τ-model. Section 3 discusses the existence of travelling waves in the PDE and its dependence on the non-equilibrium parameter τ. In Section 4 we work out the adaptive moving grid method in terms of a coordinate transformation. The numerical results showing the different scenarios are described in Section 5. The main features of the model are described by three important cases, which also can be found in the second frame of Figure 1: monotone waves, non-monotone waves with oscillations and non-monotone waves with the formation of plateaus. The appearance of each of these waves depends strongly on the parameters in the model. Finally, Section 6 summarizes the conclusions.