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Numerical Methods to Solve the Complex Symmetric Stabilizing Solution of the Complex Matrix Equation $X+A^TX^{-1}A=Q$

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Abstract. When the matrices A and Q have special structure, the structure-preserving algorithm was used to compute the stabilizing solution of the complex matrix equation $X+A^TX^{-1}A=Q$. In this paper, we study the numerical methods to solve the complex symmetric stabilizing solution of the general matrix equation $X+A^TX^{-1}A=Q$. We not only establish the global convergence for the methods under an assumption, but also show the feasibility and effectiveness of them by numerical experiments.

AMS subject classifications: 65F10, 65F35, 65F50

Key words: Complex matrix, complex symmetric stabilizing solution, fixed-point method, structure-preserving algorithm.

1 Introduction

The nonlinear matrix equation $X+A^TX^{-1}A=Q$, where A is real and Q is symmetric positive definite, arises in several applications, such as the analysis of ladder network, dynamic programming, the Green's function in nano research, control theory and stochastic filtering. These equations have been studied in [5,6], for example.

Recently, there arises the need to consider the matrix equation

$$X+A^TX^{-1}A=Q, \tag{1.1}$$

where A is complex and Q is complex symmetric. First, it is explained in [2] that the computation of the surface Green's function in nano research [7] can be reduced to the problem of solving the matrix equation (1.1), where $Q=Q_1+i\eta I$ with Q_1 real symmetric and η positive scalar, but the matrix A is still a real matrix. And then it is shown in [4] that a quadratic eigenvalue problem arising from the vibration analysis of fast trains can

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be solved efficiently and accurately by solving a matrix equation of the form (1.1), where A is complex and Q is complex symmetric. Moreover, the matrix A has only one nonzero block in the upper-right corner, and Q is block tridiagonal and block Toeplitz. In those two applications, the existence of a unique complex symmetric stabilizing solution has been proved using advanced results on linear operators. The fixed-point method and doubling algorithm were given to solve the stabilizing solution of the matrix equation (1.1).

For the more general complex equation (1.1), the existence of a unique complex symmetric stabilizing solution has been proved in [1]. However, the corresponding numerical experiments were not given. In this paper, according to the idea proposed in [1], we mainly discuss the numerical algorithms to solve the stabilizing solution of this equation. In Section 2, we introduce the preliminaries of the complex matrix equation (1.1). In Section 3, the fixed-point method (FPI), modified fixed-point method (MFPI) and structure-preserving algorithm (SPA) are proposed to find the complex symmetric stabilizing solution of (1.1) and their convergence are analyzed under an assumption. In Section 4, numerical examples are given to show the feasibility and effectiveness of the FPI, MFPI and SPA methods, and concluding remarks are made in Section 5.

2 Preliminaries

For equation (1.1) we write:

$$\begin{aligned} &A = A_1 + iA_2, \quad Q = Q_1 + iQ_2, \\ &A_1, A_2 \in \mathbb{R}^{n\times n}, \quad Q_1 = Q_1^T, \quad Q_2 = Q_2^T \in \mathbb{R}^{n\times n}. \end{aligned} \tag{2.1}$$

Definition 2.1. We define that

(a) a solution X of (1.1) is said to be stabilizing if $\rho(X^{-1}A) < 1$, where $\rho(\cdot)$ denotes the spectral radius;

(b) $W > 0$ denotes the positive definiteness of a Hermitian matrix W.

The following theorem is given by [1].

Theorem 2.1. ([1]) *If the matrices A_2 and Q_2 satisfy that*

$$Q_2 + e^{i\theta} A_2^T + e^{-i\theta} A_2 > 0, \theta \in [0, 2\pi], \tag{2.2}$$

then the equation (1.1) *has a stabilizing solution.*

We suppose the inequality (2.2) holds throughout this paper. Obviously, if a positive semi-definite matrix is added to Q_2, it still holds. Let

$$M_0 = \begin{bmatrix} A & 0 \\ Q & -I \end{bmatrix}, \quad L_0 = \begin{bmatrix} 0 & I \\ A^T & 0 \end{bmatrix}. \tag{2.3}$$