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Regularity Criteria for the Navier-Stokes Equations Containing Two Velocity Components

Hui Zhang*

School of Mathematical Science, Anqing Teacher's College, AnQing, Hunan 246011, P. R. China

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Abstract. Based on the critical sobolev inequalities in the Besov spaces with the logarithmic form, the regularity criteria in terms of two velocity components for the 3D incompressible Navier-Stokes equations is improved.

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1 Introduction

In this paper, we consider the following incompressible Navier-Stokes equations in $\mathbb{R}^3 \times (0,T)$

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \Delta u + \nabla p = 0 \\ \quad \nabla \cdot u = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

Where $u = u(x,t)$ is the velocity field, $p(x,t)$ is the scalar pressure and $u_0(x)$ with $\nabla\cdot u_0 = 0$ in the sense of distribution is the initial velocity field.

It is well known that for $u_0(x) \in L^2(\mathbb{R}^3)$, (1.1) exist at least one weak solution that is called Leray-Hopf weak solution. Nevertheless, the fundamental problem of the uniqueness and regularity of such solutions is still open.

However, the solution regularity can been derived when certain growth conditions are satisfied. This is known as a regularity criterion problem introduced in the celebrated work of Serrin [1], and can be described as follows:

*Corresponding author. *Email address:* `zhangaqtc@126.com` (H. Zhang)

A weak solution u is regular if the growth condition

$$u \in L^p(0,T;L^q(\mathbb{R}^3)); \quad \frac{2}{p}+\frac{3}{q}=1, \quad 3<q\leq\infty, \tag{1.2}$$

holds true.

There is a large literature on improvement of the condition (1.2). It may be superfluous to recall all results. To go directly to the main points of the present paper, we only review some known results which are closely related to our main result. Beirã da Veiga [2] improved the condition (1.2) in terms of two velocity components

$$\tilde{u} \in L^p(0,T;L^q(\mathbb{R}^3)); \quad \frac{2}{p}+\frac{3}{q}=1, \quad 3<q\leq\infty, \tag{1.3}$$

where $\tilde{u}=(u_1,u_2,0)$ is the horizontal velocity.

Very recently, Zhang [3] extended the condition (1.3) into BMO space in the marginal case $q=\infty$

$$\tilde{u} \in L^2(0,T;BMO). \tag{1.4}$$

Another interesting contribution of this problem is due to Beirão da Veiga [4] on the regularity criterion with respect to the velocity gradient condition

$$\nabla u \in L^p(0,T;L^q(\mathbb{R}^3)); \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2}<q\leq\infty. \tag{1.5}$$

Recently, based on the Littelewood-Paley decomposition to the equations(1.1), Dong and Zhang [5] extended the regularity criterion via two components of velocity field in homogeneous Besov space

$$\nabla_h \tilde{u} \in L^2(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)); \quad \nabla_h \tilde{u}=(\partial_1\tilde{u},\partial_2\tilde{u}). \tag{1.6}$$

Penel and Pokorny [6] obtained an improved regularity result in L^p spaces

$$\partial_1 u_1,\partial_2 u_2 \in L^p(0,T;L^q(\mathbb{R}^3)); \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2}<q\leq\infty. \tag{1.7}$$

Dong and Chen [7] improved the condition (1.7) in Lorentz space, Morrey space and multiplier space. Actually, the weak solution remains regular if the single velocity component satisfies some conditions (see [8,9,16]).

The aim of this present paper is to extend the regularity criterion (1.6) and (1.7) in homogeneous Besov space in the marginal case. More precisely, we will prove the following result.

Theorem 1.1. *Suppose $u_0 \in H^3(\mathbb{R}^3)$ and $\nabla\cdot u_0=0$ in the sense of distributions. Assume that u is a Leray-Hopf weak solutions of (1.1) on $(0,T)$. If u satisfies the following condition*

$$\int_0^T \frac{\|(\partial_1 u_1,\partial_2 u_2)\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+log(1+\|(\partial_1 u_1,\partial_2 u_2)\|_{\dot{B}^0_{\infty,\infty}})}}dt<\infty, \tag{1.8}$$

then the weak solution is regular on $(0,T]$.