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A Nonstandard Higher-Order Variational Model for Speckle Noise Removal and Thin-Structure Detection

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Abstract. We propose a multiscale approach for a nonstandard higher-order PDE based on the $p(\cdot)$-Kirchhoff energy. We first use the topological gradient approach for a semi-linear case in order to detect important objects of the image. We consider a fully nonlinear $p(\cdot)$-Kirchhoff equation with variable-exponent functions that are chosen adaptively based on the map provided by the topological gradient in order to preserve important features of the image. Then, we consider the split Bregman method for the numerical implementation of the proposed model. We compare our model with other classical variational approaches such as the TVL and bi-harmonic restoration models. Finally, we present some numerical results to illustrate the effectiveness of our approach.

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Key words: Inverse problems, regularization procedures, $p(\cdot)$-Kirchhoff, topological gradient, split Bregman.

1 Introduction

Image restoration is a fundamental task in image processing and it arises in diverse fields like geophysics, optics, medical imaging, etc. In this work, we are interested in the restoration of images highly corrupted with multiplicative noise. Such a problem is a

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challenging task in various fields and particularly in ultrasound medical imaging. The reason is that ultrasound images are strongly influenced by the quality of data usually corrupted with Rayleigh-distributed multiplicative noise. The latter is called speckle noise [40, 41, 44] and usually affects image analysis methods by making important features hard to detect. We aim at reconstructing an image $u:\Omega\to\mathbb{R}$ from an observed one $f:\Omega\subset\mathbb{R}^2\to\mathbb{R}$ which is degraded and contaminated by noise. The degradation model that we consider is the following:

$$f=u+\eta\sqrt{u}, \tag{1.1}$$

where $\eta:\Omega\to\mathbb{R}$ is a positive function that follows the Rayleigh-distribution. The reconstruction problem based on model (1.1) is well known to be an ill-posed inverse problem and thus regularization techniques are needed. Generally, the regularization technique turns the reconstruction problem based on model (1.1) into a well-posed optimization one where the energy to be minimized is the sum of a regularization term (mostly a semi-norm of a functional space fixed a priori) and a data fitting term. The well-posed minimization problem has the following form:

$$\min_{\{u>0;u\in\mathcal{H}\}}\left\{\mathcal{J}(u):=J(u)+\lambda\int_\Omega\left(\frac{f-u}{\sqrt{u}}\right)^2\mathrm{d}x\right\}. \tag{1.2}$$

The first part in the energy $\mathcal{J}(\cdot)$ is a regularization term, the second one is the fitting term, λ is a positive weight which controls the trade-off between them and $\mathcal{H}$ is the space where the solution is sought.

Motivated by the advantages of the total variation, the authors in [40] proposed a convex variational model which consists in minimizing

$$\min_{\{u>0;u\in\mathcal{H}\}}\left\{\mathcal{J}(u):=\int_\Omega|Du|+\lambda\int_\Omega\left(\frac{f-u}{\sqrt{u}}\right)^2\mathrm{d}x\right\}. \tag{1.3}$$

However, the total variation regularizer produces staircase effects in the restored images. In [58], the authors proposed the fitting term

$$\int_\Omega\left(\frac{f}{\sqrt{u}}\log\frac{f}{u}-\frac{f}{\sqrt{u}}+\sqrt{u}\right)\mathrm{d}x,$$

which is derived from the Kullback-Leibler divergence, also known as the I-divergence [45]. After that, they used first- and second-order total variation as a regularization which yields the following model:

$$\min_{u\in\mathcal{H}_1}\left\{\alpha\int_\Omega|D^2u|\mathrm{d}x+\beta\int_\Omega|Du|\mathrm{d}x+\lambda\int_\Omega\left(\frac{f}{\sqrt{u}}\log\frac{f}{u}-\frac{f}{\sqrt{u}}+\sqrt{u}\right)\mathrm{d}x\right\}, \tag{1.4}$$

where α, β and λ are regularization parameters. Instead of solving the model (1.4), they in practice used the auxiliary variable $z=\log u$ and then solved:

$$\min_{z\in\mathcal{H}_2}\left\{\alpha\int_\Omega|D^2z|\mathrm{d}x+\beta\int_\Omega|Dz|\mathrm{d}x+\lambda\int_\Omega(fe^{-z/2}\log\frac{f}{e^z}-fe^{-z/2}+e^{z/2})\mathrm{d}x\right\}. \tag{1.5}$$