J. Math. Study
doi: 10.4208/jms.v52n3.19.05

Vol. **52**, No. 3, pp. 299-319
September 2019

A New Mixed Method for the Stokes Equations Based on Stress-Velocity-Vorticity Formulation

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Received January 30, 2019; Accepted June 1, 2019

Abstract. In this paper, we develop and analyze a mixed finite element method for the Stokes flow. This method is based on a stress-velocity-vorticity formulation. A new discretization is proposed: the stress is approximated using the Raviart-Thomas elements, the velocity and the vorticity by piecewise discontinuous polynomials. It is shown that if the orders of these spaces are properly chosen then the advocated method is stable. We derive error estimates for the Stokes problem, showing optimal accuracy for both the velocity and vorticity.

AMS subject classifications: 65N12, 65N30, 76D07, 76M10

Key words: Mixed finite element, Stokes equations, Raviart-Thomas, incompressible fluids.

1 Introduction

It is hard to give a precise definition of mixed methods, see also [1], generally the definition refers to a class of methods based on the simultaneous approximations of a primal and a dual quantity: for example displacement and stress for the elastic problem, temperature and heat flux for the heat equation. To clarify the concept let us consider the simplest form of the steady heat equation:

$$-\Delta u = f,$$

where u is the temperature (primal quantity) and f is the source term. In order to obtain the mixed formulation (or dual formulation) we have a second quantity $\boldsymbol{q} = -\operatorname{grad} u$, the heat flux, that is identified as dual variable. So that the heat equation becomes:

$$\operatorname{div} \boldsymbol{q} = f.$$

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The dual formulation is written as a system of two equations, the first one is the definition of heat flux, the second one is the energy conservation law. In the same way we can identity a pair of primal and dual equation for the linear elasticity problem. The primal formulation is given by

$$-\operatorname{div}(\mathbb{C}\operatorname{grad}\boldsymbol{u})=\boldsymbol{f},$$

where $\boldsymbol{u}$ is the displacement and with $\mathbb{C}$ we denote the elasticity tensor. In the dual equation we have to introduce the stress tensor defined as $\sigma=\mathbb{C}\operatorname{grad}\boldsymbol{u}$; the problem can be written as

$$-\operatorname{div}\sigma=\boldsymbol{f},$$

which is the linear momentum conservation law. This is equivalent to the Hellinger-Reissner principle. To be more precise, we have also to enforce the conservation of angular momentum, that is given by the symmetry of σ, this is what makes the construction of a finite element method really complicated.

The usual mixed formulation of Stokes problem is given in terms of velocity and pressure as:

$$\begin{cases}-\Delta\boldsymbol{u}+\operatorname{grad}p=\boldsymbol{f},\\ \operatorname{div}\boldsymbol{u}=0.\end{cases}$$

Even though it is algebraically equivalent to the linear elasticity problem, only a few authors [2–4] contributed to the development of a proper dual formulation, or a *real* mixed formulation of Stokes equation.

Taking as reference the examples of heat equation and linear elasticity problem, we can say that the usual velocity-pressure formulation is the primal system of equations. The dual one should include at least two equations: the rheological model and the conservation of linear momentum; the unknowns should be the stress tensor and the velocity fields. One of the most attractive feature of the advocated numerical method is the fact that it is able to approximate simultaneously the stress, the velocity and the vorticity which are all physical quantities of interest in practical application. Hence no post-processing techniques (which degrades the order of accuracy) for computing the stress or the vorticity is required. An application where the proposed methodology could be employed is the poroelasticity.

The Stokes equations can be seen as a particular case of the linear elasticity problem, in which the bulk modulus tends to infinity (incompressible case). The theory can be deduced using the fact that all the theoretical results about the Hellinger-Reissner formulation are independent of the bulk modulus, in particular there are no extra hypotheses for treating the incompressible case, see [5, Section 2].

In this paper we give some novel theoretical results about the Stokes equations in their dual form without using the equivalence with the primal one: this is, to our knowledge, a new contribution that can help in giving a more complete picture of the mixed methods for fluid dynamics problems.