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Reduced-Order Modelling for the Allen-Cahn Equation Based on Scalar Auxiliary Variable Approaches

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Abstract. In this article, we study the reduced-order modelling for Allen-Cahn equation. First, a collection of phase field data, i.e., an ensemble of snapshots of at some time instances is obtained from numerical simulation using a time-space discretization. The full discretization makes use of a temporal scheme based on the scalar auxiliary variable approach and a spatial spectral Galerkin method. It is shown that the time stepping scheme is unconditionally stable. Then a reduced order method is developed using by proper orthogonal decomposition (POD) and discrete empirical interpolation method (DEIM). It is well-known that the Allen-Cahn equations have a nonlinear stability property, i.e., the free-energy functional decreases with respect to time. Our numerical experiments show that the discretized Allen-Cahn system resulting from the POD-DEIM method inherits this favorable property by using the scalar auxiliary variable approach. A few numerical results are presented to illustrate the performance of the proposed reduced order method. In particular, the numerical results show that the computational efficiency is significantly enhanced as compared to directly solving the full order system.

AMS subject classifications: 76T10, 78M34, 74S25
Key words: Allen-Cahn equation, scalar auxiliary variable, proper orthogonal decomposition, discrete empirical interpolation method.

1 Introduction and motivations

Reduced-Order Model (ROM) applied to numerical design in modern engineering is a tool that is wide-spreading in the scientific community. It is particularly useful in solv-

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ing complex realistic multi-parameters, multi-physics and multi-scale problems, where classical methods such as Finite Difference, Finite Element, and Finite Volume methods would require up to billions of unknowns. On the contrary, ROM is based on a sharp offline/online strategy, which can be realized with a reduced number of unknowns. Such a strategy can be used to handle control, optimization, prediction, and data analysis problems in almost real-time, that is, ultimately, a major goal for industrials. The reduced order modeling offline strategy relies on proper choices for data sampling and construction of the reduced basis, which will be used then in the online phase, where a proper choice of the reduced model describing the dynamic of the system is needed. The key feature of ROM is its capability to drastically reduce the computational cost of numerical simulations, and thus highly speedup computations without compromising too much the physical accuracy of the solution from the engineering point of view. Among the most popular ROM approaches, Proper Orthogonal Decomposition (POD) strategy provides optimal (from the energetic point of view) basis or modes to represent the dynamics from a given database (snapshots) obtained by a full-order system. Onto these reduced basis, a Galerkin projection of the governing equations can be employed to obtain a low-order dynamical system for the basis coefficients. The resulting low-order model is named standard POD-ROM, which thus consists in the projection of high-fidelity (full-order) representations of physical problems onto low-dimensional spaces of solutions, with a dramatically reduced dimension. The main advantage of POD-ROM is that these low-dimensional spaces are capable of capturing the dominant characteristics of the solution, and the computations in the low-dimensional space can be done at a reduced cost. This advantage has led researchers to apply POD to a variety of physical and engineering problems, including the Navier Stokes equations in computational fluid dynamics; See e.g. [2–4] and [8].

We aim in this paper at applying this reduced-order model strategy to solve the Allen-Cahn equation and investigating its efficiency in terms of stability, convergence, and data reduction.

The Allen-Cahn equation was originally introduced to describe the motion of anti-phase boundaries in crystalline solids [1], and has now been used to model many moving interface problems from fluid dynamics to materials science via a phase-field approach. It consists in finding $\phi:\Omega\times(0,T]\to\mathbb{R}$ solution of

$$\begin{cases} \frac{\partial\phi}{\partial t}+\gamma\big(-\Delta\phi+f(\phi)\big)=0, & \forall(\boldsymbol{x},t)\in\Omega\times(0,T],\\ \nabla\phi\cdot\boldsymbol{n}\big|_{\partial\Omega}=0, & \forall t\in(0,T],\\ \phi(t=0)=\phi^0(\boldsymbol{x}), & \forall\boldsymbol{x}\in\Omega. \end{cases} \tag{1.1}$$

In the above, γ is a positive kinetic coefficient, $\Omega\subset\mathbb{R}^d$ is a bounded domain, $\boldsymbol{n}$ is the outward normal, $f(\phi)=F'(\phi)$ with the given function $F(\phi)=\frac{1}{4\epsilon^2}(\phi^2-1)^2$ being the Ginzburg-