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IMPROVED PMHSS ITERATION METHODS FOR COMPLEX SYMMETRIC LINEAR SYSTEMS*

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Abstract

Based on the preconditioned modified Hermitian and skew-Hermitian splitting (PMHSS) iteration method for the complex symmetric linear system, two improved iterative methods, namely, the modified PMHSS (MPMHSS) method and the double modified PMHSS (DMPMHSS) method, are proposed in this paper. The spectral radii of the iteration matrices of two methods are given. We show that by choosing an appropriate parameter, MPMHSS could speed up the convergence on PMHSS. The DMPMHSS method is a four-step alternating iteration that is developed upon the two-step alternating iteration of MPMHSS. We discuss the choice of the parameters and establish the convergence of DMPMHSS. In particular, we give an analysis of the spectral radius of PMHSS and DMPMHSS at the parameter free situation, and we show that DMPMHSS converges faster than PMHSS in most cases. Our numerical experiments show these points.

Mathematics subject classification: 65F10, 65N22.
Key words: Complex symmetric linear system, PMHSS.

1. Introduction

In this paper, we are concerned with iterative methods for the following linear system

$$Ax = b, \qquad A \in C^{n\times n},\ x, b \in C^n, \tag{1.1}$$

where A is a complex symmetric matrix of the form

$$A = W + iT,$$

and $W, T \in R^{n\times n}$ are real symmetric matrices, with W being positive definite and T positive semidefinite. Here and in the sequel, we use $i = \sqrt{-1}$ to denote the imaginary unit. We assume $T \neq 0$, which implies that A is non-Hermitian.

Let $x = y + iz$, $b = p + iq$. Then the real form of (1.1) is equivalent to

$$\begin{bmatrix} W & -T \\ T & W \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \tag{1.2}$$

and this paper considers efficient iterative methods for this real form (1.2). Theoretical analyses and computational results show that reformulating a complex linear system into the equivalent real form (1.2) is a feasible and effective approach, for which we can construct, analyze, and implement accurate, efficient, and robust preconditioned iteration methods [3].

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Complex symmetric linear systems (1.1) arise in many problems in scientific computing and engineering applications. Examples include the diffuse optical tomography [1], FFT-based solution of certain time-dependent PDEs [12], quantum mechanics [18], molecular scattering [17], structural dynamics [13], distributed control problems [6], and lattice quantum chromodynamics [15]. For more examples and references, we refer to [11].

In these applications, these matrices are very large and are usually sparse. The Krylov subspace method is a very natural choice. However, since A is not a Hermitian matrix, the classical CG algorithm can not be applied. The simplest way is to use the preconditioned CG algorithm to solve $A^H Ax = A^H b$, or to transform (1.1) into a real 2n dimension situation (1.2) that can be solved by some Krylov subspace iteration methods, such as CGS, TFQMR , BiCGstab(l), GMRES. But these methods can be used for any linear system without using the special structure of the matrix A and need more storage requirements or computational costs.

Apart from the Krylov subspace methods, some real valued iterative algorithms are also proposed to solve complex symmetric linear systems. For more details, please refer to [2].

Based on the Hermitian and skew-Hermitian splitting in the generalized Lanczos method [20], Bai et al. proposed the Hermitian and skew-Hermitian splitting (HSS) [9] method which is very efficient and unconditional convergence, the preconditioned HSS (PHSS) [8] method, and the accelerated HSS (AHSS) [7] method which is faster than PHSS method. Based on the AHSS method, [16] proposed the preconditioned AHSS (PAHSS) method. They all can be used to solve this complex symmetric linear systems. But a common potential difficulty with these iteration approachs needs to solve the shifted skew-Hermitian sub-system of linear equations at each iteration step. In many cases, it is as difficult to solve these sub-systems as the original problem. For more details, please refer to [4]. A modification of the HSS (MHSS) iteration scheme for complex symmetric linear systems is proposed by Bai et al [4], and only two linear sub-systems with real and symmetric positive definite coefficient matrices need to be solved at each step. We can know from [4] that the MHSS method is convergent for any positive parameter α and the convergence speed (also the spectral radius of the iteration matrix) of the MHSS iteration is bounded by

$$\max_{\lambda_i \in \lambda(W)} \frac{\sqrt{\alpha^2 + \lambda_i^2}}{\alpha + \lambda_i} \cdot \max_{\lambda_i \in \lambda(T)} \frac{\sqrt{\alpha^2 + \lambda_i^2}}{\alpha + \lambda_i}.$$

To further generalize the MHSS method and accelerate its convergence rate, Bai et al. [5, 6] propose a preconditioned MHSS (PMHSS) method for solving the complex symmetric linear system (1.2). The iteration scheme is as follow:

$$\left\{ \begin{array}{l} \begin{bmatrix} \alpha V + W & 0 \\ 0 & \alpha V + W \end{bmatrix} \begin{bmatrix} y^{(k+\frac{1}{2})} \\ z^{(k+\frac{1}{2})} \end{bmatrix} = \begin{bmatrix} \alpha V & T \\ -T & \alpha V \end{bmatrix} \begin{bmatrix} y^{(k)} \\ z^{(k)} \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix}, \\ \begin{bmatrix} \alpha V + T & 0 \\ 0 & \alpha V + T \end{bmatrix} \begin{bmatrix} y^{(k+1)} \\ z^{(k+1)} \end{bmatrix} = \begin{bmatrix} \alpha V & -W \\ W & \alpha V \end{bmatrix} \begin{bmatrix} y^{(k+\frac{1}{2})} \\ z^{(k+\frac{1}{2})} \end{bmatrix} + \begin{bmatrix} q \\ -p \end{bmatrix}. \end{array} \right. \tag{1.3}$$

For arbitrary initial values, it is proved [5, 6] that the iteration converges to the unique solution of the linear system (1.2) for arbitrary parameter $\alpha > 0$, and moreover, the spectral radius of the iteration matrix is bounded by

$$\sigma(\alpha) = \max_{\tilde{\lambda}_j \in \lambda(V^{-1}W)} \frac{\sqrt{\alpha^2 + \tilde{\lambda}_j^2}}{\alpha + \tilde{\lambda}_j} \cdot \max_{\tilde{\mu}_j \in \lambda(V^{-1}T)} \frac{\sqrt{\alpha^2 + \tilde{\mu}_j^2}}{\alpha + \tilde{\mu}_j} \tag{1.4}$$