Journal of Computational Mathematics
Vol.37, No.2, 2019, 240–260.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1712-m2017-0064

A BLOCK LANCZOS METHOD FOR THE CDT SUBPROBLEM*

Liqiang Song and Wei Hong Yang
School of Mathematical Sciences, Fudan University, Shanghai 200433, China
Email: lqsong14@fudan.edu.cn, whyang@fudan.edu.cn

Abstract

In this paper, we present a block Lanczos method for solving the large-scale CDT subproblem. During the algorithm, the original CDT subproblem is projected to a small-scale one, and then some classical method is employed to solve this small-scale CDT subproblem to get a solution, which can be used to derive an approximate solution of the original CDT subproblem. Theoretical analysis of the error bounds for both the optimal value and the optimal solution is also proposed. Numerical experiments are carried out, and it is demonstrated that the block Lanczos method is effective and can achieve high accuracy for large-scale CDT subproblems.

Mathematics subject classification: 90C20, 90C30, 65K05.
Key words: CDT subproblem, Trust-region subproblem, Block-Lanczos method, Krylov subspace, Error bounds.

1. Introduction

Recently several research works have focused on the Celis-Dennis-Tapia (CDT) subproblem; see [3–5, 21, 25]. The CDT subproblem has the following form:

$$\min_{d\in\mathbb{R}^n} \quad \Phi(d) \equiv \frac{1}{2}d^T Bd + g^T d \tag{1.1}$$

$$\text{s.t.} \quad \|d\|_2 \leq \Delta \tag{1.2}$$

$$\|A^T d + c\|_2 \leq \xi, \tag{1.3}$$

where $B \in \mathbb{R}^{n\times n}$ is symmetric, $A \in \mathbb{R}^{n\times m}$, $g \in \mathbb{R}^n$, $c \in \mathbb{R}^m$, $\Delta > 0$ and $\xi \geq 0$. Problem (1.1)-(1.3) was first proposed in [6], and later it was also used in [18] for equality constrained optimization to achieve global convergence. Since then, many research works have been devoted to the theoretical properties of the CDT subproblem; see, e.g. [1,3,4,7–9,11,14,23,29]. A related problem is the quadratic optimization with two quadratic constraints, which includes the CDT subproblem as a special case. Theoretical results on this topic are also discussed by many authors; see [2, 17, 28].

There have been various numerical methods for solving the CDT subproblem. The approach based on the dual function of the CDT subproblem (1.1)-(1.3) is successfully used in [13, 30, 31]. Another powerful tool for the CDT subproblem is the semidefinite programming (SDP) relaxation method. Many interesting works on this topic can be found in [2, 5, 25] and the references therein. Recently, a new algorithm is proposed by Sakaue et al. [21]. Roughly speaking, this method computes all the Lagrange multipliers of the problem by solving a linear

* Received March 17, 2017 / Revised version received September 4, 2017 / Accepted December 18, 2017 / Published online August 14, 2018 /

eigenvalue problem, obtains the corresponding KKT points, and finds a global solution as the KKT point with the smallest objective value.

In this paper, we take a different approach from the above-mentioned methods, and consider a subspace method for large-scale CDT subproblems. Specifically, we propose a Krylov subspace method to project the CDT subproblem (1.1)-(1.3) to a small-size CDT subproblem

$$\min_{x\in\mathbb{R}^{k(m+1)}} \quad \widetilde{\Phi}(x) \equiv \frac{1}{2}x^T T_k x + \widetilde{g}^T x \tag{1.4}$$

$$s.t. \quad \|x\|_2 \le \Delta \tag{1.5}$$

$$\|\widetilde{A}^T x + c\|_2 \le \xi, \tag{1.6}$$

where $T_k = Q_k^T B Q_k$, $\widetilde{g} = Q_k^T g$, $\widetilde{A} = Q_k^T A$ for some orthonormal basis matrix $Q_k \in \mathbb{R}^{n\times k(m+1)}$. Then we solve the problem (1.4)-(1.6) by some classical methods to obtain an optimal solution x^*. Specifically, if T_k is positive definite, Yuan's dual algorithm [30] is employed in our approach for (1.4)-(1.6); When T_k is indefinite, we use the SDP relaxation method to solve (1.4)-(1.6). The approximate solution of (1.1)-(1.3) is set to be $d_k = Q_k x^*$. Numerical results demonstrate that the optimal solution of the original CDT subproblem can be accurately approximated by d_k. This is an expected result since Krylov subspace method has been proved to be successful in optimization problems. The readers who are interested in the topic of Krylov subspace optimization methods can refer to [10, 15].

This paper is organized as follows. In section 2, we present some notations, definitions and preliminary results used in this paper. The block Lanczos method for the CDT subproblem is given in detail in section 3. Error bounds on the optimal value are presented in section 4, and the bounds on the optimal solution are established in section 5. Some numerical experiments comparing the block Lanczos method with other state-of-the-art algorithms are presented in section 6. The paper ends with some conclusions and a short discussion on possible future work.

2. Notations and Preliminary Results

In this section, we introduce the notations, definitions, and preliminary results which will be used throughout the paper. As usual, I_n denotes the $n \times n$ identity matrix, e_i denotes the i-th column of I_n. We use E^n_{m+} to denote the first m columns of I_n, and E^n_{m-} to denote the last m columns of I_n. The notation span$\{v_1, v_2 \cdots, v_k\}$ denotes the linear space spanned by vectors $v_1, v_2 \cdots, v_k$. For a matrix A, we use $\mathcal{R}(A)$ and $\mathcal{N}(A)$ to denote the range and the kernel of A. To simplify our presentation, we shall also adopt MATLAB-like convention to access the entries of vectors and matrices. For example, $(i : j)$ stands for the set of integers from i to j inclusive, and $A(k : l, i : j)$ is the submatrix of A that consists of intersections of row k to row l and column i to column j.

For the CDT subproblem (1.1)-(1.3), let

$$\xi_{\min} := \min_{\|d\|_2\le\Delta} \|A^T d + c\|_2. \tag{2.1}$$

The following results tell us that $\xi > \xi_{\min}$ is the only interesting case. Thus, in the remaining of the paper, we assume that $\xi > \xi_{\min}$.

Lemma 2.1 ([30, Lemma 2.2]). *The following statements hold:*

1) *if* $\xi = \xi_{\min}$, *then either there is only one feasible solution of* (1.2)-(1.3), *or* (1.1)-(1.3) *can be reduced to a simpler problem which has the form of* (1.1)-(1.2);